Flow models of distributed computations : event structures and nets

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UNITÉ DE RECHERCHE
INRIA-SOPHIA ANTIPOLIS

Rapports de Recherche

N° 1482

Programme 2
Calcul Symbolique, Programmation et Génie logiciel

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Juillet 1991

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Flow Models of Distributed Computations:

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Structures d'événements et réseaux à flux

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Abstract.

We introduce flow event structures, which provide an intermediate between prime and stable event structures. These three kinds of event structures are abstractly equivalent in that they give concrete presentations for the same kind of domains, namely Berry's dI-domains. However, regarded as means of defining families of configurations, these event structures are increasingly more powerful. From this point of view, flow event structures are related to a new kind of Petri nets, which we call flow nets: we show that for each flow event structure we can build a flow net having the same configurations, a configuration of a net being a set of events firable in sequence, and conversely. This generalizes the result of Nielsen, Plotkin and Winskel relating prime event structures and occurrence nets. Moreover we show that safe flow nets correspond to the bundle event structures introduced recently by Langerak. This establishes a strict hierarchy upon the various kinds of event structures, prime, bundle, flow and stable ones.

Résumé.

Nous introduisons les structures d'événements "à flux", qui sont intermédiaires entre les structures premières et stables. Ces trois types de structures d'événements ne se distinguent pas d'un point de vue abstrait, étant autant de manières de présenter concrètement la même classe de domaines, précisément: la classe des dI-domaines de Berry. Mais ces trois types s'ordonnent si on les compare du point de vue des familles de configurations engendrées. Nous montrons que de ce point de vue les structures d'événements "à flux" correspondent à une nouvelle sorte de réseaux de Petri, que nous appelons les réseaux "à flux": pour chaque structure d'événements à flux nous pouvons construire un réseau à flux qui a les mêmes configurations – les configurations d'un réseau étant les ensembles d'événements franchissables en séquence –, et inversement. Ce résultat généralise celui de Nielsen, Plotkin et Winskel, qui reliait les structures d'événements premières aux réseaux d'occurrences. De plus nous montrons que les réseaux à flux saufs correspondent aux structures d'événements "en botte" de Langerak. Ainsi nous montrons qu'il existe une hiérarchie stricte entre ces différents types de structures d'événements.

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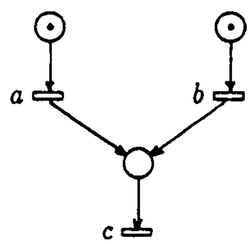
1. Introduction.

In this paper we investigate the relationships between two models for concurrent computations, namely Petri nets and Winskel's event structures. A distinctive feature of these models, with respect to transition systems used in the so-called interleaving semantics, is that they provide an adequate account of the *causal* relations between events in a distributed system. At least this is true for event structures, if not as clear for Petri nets. For instance in the net



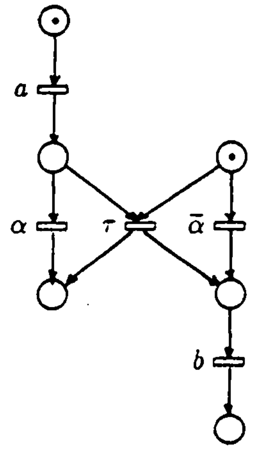
it is not clear whether a and b, when they both occur, are causes of c; this depends on the token that was chosen to fire c. This is a typical example of "non-stable" net. Nevertheless, for some classes of nets, like condition/event systems, one has a definite notion of causality. This is also true for the *occurrence nets* introduced by Nielsen, Plotkin and Winskel in their pioneering paper [15], where a close connection between occurrence nets, *prime event structures* and prime algebraic coherent domains was established. Our aim is to generalize their work, so as to be able to compare various semantics for process description languages.

Petri nets have been used to provide a "truly concurrent" semantics for CCS-like languages [14], for instance by Goltz [10,12], Degano, De Nicola and Montanari [7], Olderog [16], van

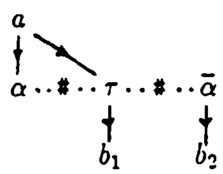
This work has been partly supported by ESPRIT Basic Research Action 3011 CEDISYS.

Glabbeek and Vaandrager [9], and Taubner [17]. Similarly, event structure semantics for such languages have been proposed, by Winskel [19,20] using *stable* event structures, and by Goltz [11], Degano, De Nicola and Montanari [8], Vaandrager [18] using *prime* event structures. As a matter of fact, such a semantics also yields a Petri net semantics, at least indirectly by means of the domains of configurations. On the other hand, it seems difficult to relate a net semantics of CCS with an event structure semantics of the same language, unless the nets used are occurrence nets - but these, just like prime event structures, are quite awkward for this purpose.

An advantage of Winskel's semantics, by means of stable structures, is that the events have a clear operational meaning: they are just occurrences of actions in the syntactic tree obtained by unfolding the CCS term. For instance in the term $(a\alpha \mid \bar{\alpha}b)$ the action b corresponds to a unique event, which can be caused in two incompatible ways, either by the action $\bar{\alpha}$ alone, or by the communication action. Stable event structures account for such conflicting causes. In the same spirit, this term can be represented by the net

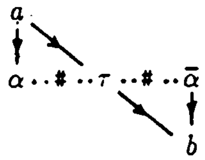


where each possible action of the term is represented by a single event. However this is not an occurrence net, and we do not know how to "extract" it out of a stable event structure, since we do not know which class of nets could correspond to stable event structures. We would like to have a concrete correspondence, preserving the families of configurations - hence in particular the names of the events -, rather than an abstract one which only preserves the domain of configurations, up to isomorphism. Note that the prime event structure semantics of $(a\alpha \mid \bar{\alpha}b)$ gives



where there are two events b_1 and b_2 representing the possible histories of the same "event" b. Using prime event structures or occurrence nets to interpret other simple terms may be quite painful - try for instance $(\alpha a\beta \mid \bar{\beta}b\bar{\alpha})$ where causality cycles arise.

To moderate these difficulties, we proposed in previous work [1,2] to deal with a more flexible notion of event structure, obtained by relaxing the axiom of conflict heredity. This allowed us to give a "natural" semantics for a CCS-like language. The language was quite restricted, however, since we did not give an event structure semantics for communication. In [3] we introduced another kind of event structures, called *flow event structures*, still similar to prime event structures, but far more relaxed – the price to pay is that the notion of configuration is not so simple as for prime event structures. In a flow event structure the causality ordering is replaced by an irreflexive *flow* relation, which is much like the existence of a place between two events in a Petri net. Moreover there is *no* requirement on the relationships between the flow and the symmetric conflict. With these structures it is fairly easy to interpret CCS terms in a "natural" way. Moreover this flow event structure semantics of CCS has an operational content: we can show that it corresponds to a "truly concurrent" semantics extracted from the usual operational semantics (see [6,3,4,5]). For instance the term $(a\alpha \mid \bar{\alpha}b)$ is now interpreted as the following flow event structure:



For the "flow" interpretation of $(\alpha a\beta \mid \bar{\beta}b\bar{\alpha})$, see [5]. Here we should also mention that all kinds of event structures – prime, flow, stable (with binary conflict) – are "abstractly" equivalent since they are concrete presentations of the same kind of domains.

In this paper we establish a preliminary result for relating various semantics of CCS-like languages: we show that flow event structures concretely correspond to a new particular class of Petri nets, which we call *flow nets*. These nets have a semantical definition, i.e. by means of firing sequences, but we show that any flow net is *equivalent* – in the sense of having the same firing sequences – to what we call a *regular* flow net. The regular flow nets are characterized by structural properties, namely:

(1) in a regular flow net a place is of one of the following two kinds:

  (1.1) a *choice* place, which can be forward branched – that is precondition of several events – but cannot be postcondition of any event;

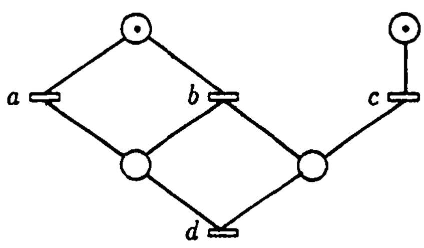
  (1.2) a *causal* place, which can be backward branched, but is precondition of at most one event.

(2) only the choice places can be initially marked.

There is also an additional requirement on causality between events:

(3) if a is a potential cause of b – that is there is a place between a and b – then there is a causal place p between a and b such that any other pre-event e of p shares a precondition with a – that is e is in direct conflict with a.

This last property ensures that in any firing sequence the causal relations between events are unambiguously determined. Therefore the flow nets are "causal" nets, although we cannot claim that any "causal" net is a flow net: there are some (stable) nets where the causality is well-defined but which are not flow nets, like for instance



We shall see that a flow net is not necessarily equivalent to a safe flow net. We will show that safe flow nets are equivalent to what we call *s-regular* flow nets. These are defined as the nets satisfying (1) and (2) above, and, instead of (3):

(3′) if two distinct events share a postcondition (causal place) then they share a precondition (choice place), hence they are in direct conflict.

It turns out that the safe flow nets concretely correspond to a new kind of event structures recently introduced by Langerak [13], the *bundle event structures.* Then our main results, establishing a concrete connection between flow event structures (resp. bundle event structures) and flow nets (resp. safe flow nets), can be stated as follows:

(1) for any flow event structure (resp. bundle event structure) we can build a flow net (resp. safe flow net) such that the sets of events firable in sequence in the net are exactly the configurations of the event structure;

(2) conversely for any flow net (resp. safe flow net) we can build a flow event structure (resp. bundle event structure) whose configurations are exactly the sets of events firable in sequence in the net.

These results generalize Nielsen, Plotkin and Winskel's one, since any occurrence net is a (safe) flow net – but not conversely –, and any prime event structure is a flow event structure – but not conversely. Summarizing, if we regard an event structure as a means to define a family of configurations, we can compare the expressive power of the various kinds of event structures (prime, bundle, flow, stable) as follows:

$$\text{PES} \subsetneq \text{BES} \subsetneq \text{FES} \subsetneq \text{SES}$$

Correspondingly, we get a hierarchy relating occurrence nets, safe flow nets and flow nets. We shall see that the inclusions above are all *strict*: for instance there is a flow event structure whose family of configurations cannot be determined by a bundle event structure.

2. Event Structures.

2.1 Domains and Prime Event Structures.

We briefly recall in this section some definitions about partially ordered sets (*cf.* [15] and [20]). Let $(D, \leq)$ be a poset. Then

- two elements x and y of D are *compatible*, in notation $x \uparrow y$, if they have an upper bound:

$$x \uparrow y \Leftrightarrow \exists z \in D \; x \leq z \;\&\; y \leq z$$

- a subset X of D is *pairwise consistent*, in notation $X\Uparrow$, if every two elements of X are compatible (in D):

$$X\Uparrow \Leftrightarrow \forall x, y \in X \; x \uparrow y$$

- the poset $(D, \leq)$ is *coherent* if every pairwise consistent subset X of D has a least upper bound $\bigsqcup X$;
- a point x in D is a *complete prime* if for any subset X of D which has a lub $\bigsqcup X$ we have:

$$x \leq \bigsqcup X \Rightarrow \exists y \in X \; x \leq y$$

  We shall denote the set of complete primes of the poset $(D, \leq)$ by $Pr(D, \leq)$ – or simply $Pr(D)$ if there is no ambiguity;

- the poset $(D,\leq)$ is *ω-prime algebraic* if $Pr(D)$ is denumerable(†) and every point of D is the lub of the complete primes it dominates, that is:

$$\forall x\in D\ x=\bigsqcup\{y\mid y\in Pr(D)\ \&\ y\leq x\}$$

  We shall denote $\hat{x}$ the set $\{y\mid y\in Pr(D)\ \&\ y\leq x\}$.

- a point x of D is *finite* if it only dominates a finite number of points, that is if the set $\{y\mid y\leq x\}$ is finite. We shall denote the set of finite points of $(D,\leq)$ by $F(D,\leq)$, or simply $F(D)$;

- the poset $(D,\leq)$ is *finitary* if each of its complete prime is finite, that is $Pr(D)\subseteq F(D)$.

The only kind of domain we shall deal with is the following:

DEFINITION (DOMAINS). *A domain is a coherent, ω-prime algebraic and finitary poset.*

Clearly a domain has a least element $\perp=\bigsqcup\emptyset$, since the empty set is pairwise consistent, and obviously $X\subseteq Y\ \Rightarrow\ \bigsqcup X\leq\bigsqcup Y$. Note that $\perp$ is not complete prime. One should note that every non empty subset X of a domain D has a greatest lower bound, namely $\bigsqcap X=\bigsqcup\{x\mid\forall y\in X\ x\leq y\}$ (the set $\{x\mid\forall y\in X\ x\leq y\}$ is pairwise consistent since $X\neq\emptyset$). It should also be obvious that if $(D,\leq)$ is a domain, then a subset X of D has a least upper bound if and only if it is bounded (that is $\exists y\in D\ \forall x\in X\ x\leq y$); a bounded set is also called *consistent*.

In the second part of the paper we shall use the fact that the whole structure of a domain $(D,\leq)$ is actually already present in the poset $(F(D),\leq)$ – where we still denote $\leq$ the restriction of the ordering of D on $F(D)$. To see this let us first show that a domain $(D,\leq)$ is isomorphic to the *ideal completion* of $(F(D),\leq)$. An *ideal* of a poset $(F,\leq)$ is a non-empty subset X of F such that:

- X is *directed*, that is:

$$x,y\in X\ \Rightarrow\ \exists z\in X\ x\leq z\ \&\ y\leq z$$

- X is a *cone* (or a left-closed or downward-closed subset) of F, that is:

$$x\in X\ \&\ y\leq x\ \Rightarrow\ y\in X$$

We denote by $\mathcal{I}(F,\leq)$, or simply $\mathcal{I}(F)$, the set of ideals of the poset $(F,\leq)$. Then the *ideal completion* $(F,\leq)^\infty$ of $(F,\leq)$ is the poset of its ideals, ordered by inclusion, that is $(F,\leq)^\infty=(\mathcal{I}(F),\subseteq)$.

We shall denote by $(D,\leq)\rightleftharpoons(D',\leq')$ the fact that two posets are isomorphic; more precisely, we shall use the notation $(D,\leq)\underset{\psi}{\overset{\varphi}{\rightleftharpoons}}(D',\leq')$ to mean that $\varphi\colon D\to D'$ and $\psi\colon D'\to D$ are two inverse poset morphisms.

LEMMA 2.1. *Any domain $(D,\leq)$ is isomorphic to the ideal completion of its poset of finite points: $(D,\leq)\underset{h}{\overset{f}{\rightleftharpoons}}(F(D),\leq)^\infty$ where f and h are given by $f(x)=\{y\mid y\in F(D)\ \&\ y\leq x\}$ and $h(X)=\bigsqcup X$.*

PROOF: let us first show that f and h are well defined: it is obvious that if X is an ideal of $F(D)$ then $\bigsqcup X$ exists, since a directed set is pairwise consistent. It is also clear that for all $x\in D$ the set $f(x)=\{y\mid y\in F(D)\ \&\ y\leq x\}$ is a cone of $F(D)$; moreover $f(x)$ is non-empty since $\perp\in f(x)$. Let

(†) a set A is denumerable if it is empty or if there exists an enumeration of A, that is a surjective mapping a from the set of positive integers onto A; in that case we can write $A=\{a_1,\ldots,a_n,\ldots\}$.

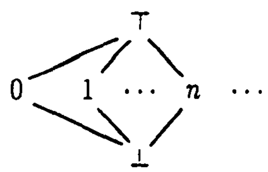
us show that $f(x)$ is directed: it is enough to prove

$$x, y \in F(D) \ \&\ x \uparrow y \ \Rightarrow\ x \sqcup y \in F(D)$$

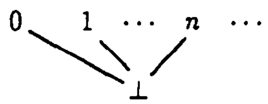
Let $u = x \sqcup y$; if $z \in Pr(D)$ is such that $z \leq u$ then we have either $z \leq x$ or $z \leq y$, therefore $\hat{u} = \hat{x} \cup \hat{y}$. Moreover since x and y are finite the set $\hat{u}$ is finite. If $z \leq u$ we have $\hat{z} \subseteq \hat{u}$, and $z = \bigsqcup \hat{z}$ since $(D, \leq)$ is ω-prime algebraic. This shows that there are only finitely many points z such that $z \leq u$, since there are only finitely many subsets of $\hat{u}$; hence $u \in F(D)$.

Since $(D, \leq)$ is finitary we have $\hat{x} \subseteq f(x)$, and since $(D, \leq)$ is ω-prime algebraic we have $x = h(f(x))$ for any $x \in D$ (for $x = \bigsqcup \hat{x} \leq \bigsqcup f(x) \leq x$). Then it is easy to see that $x \leq y \Leftrightarrow f(x) \subseteq f(y)$. It should also be obvious that if $X \in \mathcal{I}(F(D))$ we have $X \subseteq f(h(X))$. Now let $X \in \mathcal{I}(F(D))$ and $x \in F(D)$ be such that $x \leq \bigsqcup X$; then for all $z \in \hat{x}$ there exists $y \in X$ such that $z \leq y$, since z is a complete prime such that $z \leq \bigsqcup X$, hence $z \in X$ since X is a cone. Therefore we have $\hat{x} \subseteq X$; since $\hat{x}$ is finite and X is directed, there exists $y \in X$ such that $z \in \hat{x} \ \Rightarrow\ z \leq y$, hence $\bigsqcup \hat{x} \leq y$. Then $x \in X$, since $x = \bigsqcup \hat{x}$ and X is a cone. This shows $f(h(X)) = X$ for any $X \in \mathcal{I}(F(D))$ $\square$

Let us see an example of poset, which is not prime algebraic, for which the lemma does not hold; let $(D, \leq)$ be given by



We have $Pr(D) = \emptyset$, and $(F(D), \leq)$ is given by



Here the ideal completion of $F(D)$ gives $F(D)$ again. A consequence of the previous lemma is that a domain is fully determined by its poset of finite points:

COROLLARY 2.2. *Two domains* $(D, \leq)$ *and* $(D', \leq')$ *are isomorphic if and only if their posets of finite points* $(F(D), \leq)$ *and* $(F(D'), \leq')$ *are isomorphic.*

We just saw that a domain is represented as the poset of ideals of its finite points – a *representation* of a poset $(D, \leq)$ is an isomorphism from $(D, \leq)$ to $(\mathcal{F}, \subseteq)$ where $\mathcal{F}$ is a family of subsets of some given set E (i.e. $X \in \mathcal{F} \ \Rightarrow\ X \subseteq E$), ordered by inclusion. There is another way to represent a domain $(D, \leq)$ by means of a family over the set of its complete primes. To see this point, let us first observe that for any $x \in D$ the set $\hat{x} \subseteq Pr(D)$ has the following properties:

- $\hat{x}$ is pairwise consistent,
- $\hat{x}$ is a cone of $(Pr(D), \leq)$, where we still denote $\leq$ the restriction of the ordering to $Pr(D)$.

Moreover we have $x \leq y \ \Leftrightarrow\ \hat{x} \subseteq \hat{y}$. Then one may wonder whether $(D, \leq)$ is isomorphic to $(\mathcal{F}, \subseteq)$ where $\mathcal{F}$ is the family of pairwise consistent cones of $(Pr(D), \leq)$. However this cannot be quite true: there is something missing in the poset $(Pr(D), \leq)$, namely the compatibility relation of complete primes in $(D, \leq)$. For instance the two domains

$$D_0 = \begin{tikzcd} a \arrow[dr, dash] & & b \arrow[dl, dash] \\ & \bot & \end{tikzcd} \qquad D_1 = \begin{tikzcd} & \top & \\ c \arrow[ur, dash] \arrow[dr, dash] & & d \arrow[ul, dash] \arrow[dl, dash] \\ & \bot & \end{tikzcd}$$

are such that $(Pr(D_0), \leq_0) \rightleftharpoons (\{e_0, e_1\}, \emptyset) \rightleftharpoons (Pr(D_1), \leq_1)$, and the families of pairwise consistent cones of these posets are isomorphic to D_0 itself. In order to get D_1 from $(Pr(D_1), \leq_1)$, we should have kept track of the fact that $c \uparrow d$, while a and b are incompatible in D_0, in notation $a \# b$. Note that this incompatibility relation is irreflexive (that is $x \# y \;\Rightarrow\; x \neq y$), symmetric, and satisfies

$$x \# y \;\&\; y \leq z \;\Rightarrow\; x \# z$$

for $x \uparrow z \;\&\; y \leq z \;\Rightarrow\; x \uparrow y$. This motivates the following definition ([15]):

DEFINITION (PRIME EVENT STRUCTURES). *A prime event structure is a structure* $S = (E, \#, <)$ *where*

- E *is the denumerable set of events,*
- $\# \subseteq E \times E$ *is an irreflexive and symmetric relation, the conflict (or incompatibility) relation,*
- $< \subseteq E \times E$ *is a strict ordering, that is an irreflexive and transitive relation,*

satisfying:

(i) *the finite causes property: for any* $e \in E$ *the set* $\{e' \mid e' < e\}$ *is finite*

(ii) *the conflict heredity property:* $e \# e' \;\&\; e' < e'' \;\Rightarrow\; e \# e''$.

It should be now obvious that we can associate with any domain $(D, \leq)$ a prime event structure, namely $\mathcal{E}(D, \leq) = (Pr(D), \#, <)$ where $\#$ is the incompatibility relation of complete primes in the domain and $<$ the strict ordering determined by $\leq$ on $Pr(D)$. Note that a subset of $Pr(D)$ is pairwise consistent if and only if it is *conflict-free*; for any prime event structure $S = (E, \#, <)$ we shall denote by Con_S the set of conflict-free subsets of E, that is:

$$X \in Con_S \;\Leftrightarrow_{\mathrm{def}}\; e, e' \in X \;\Rightarrow\; \neg(e \# e')$$

The next step towards the representation result consists in defining the family of subsets of E determined by the prime event structure $S = (E, \#, <)$, which are called the *configurations* of the structure ([15]):

DEFINITION (CONFIGURATIONS). *Let* $S = (E, \#, <)$ *be a prime event structure. A configuration of* S *is a subset* X *of* E *such that:*

(i) X *is conflict-free:* $X \in Con_S$

(ii) X is a cone: $e \in X \;\&\; e' < e \;\Rightarrow\; e' \in X$.

We shall denote by $\mathcal{F}^\infty(S)$ the set of configurations of the prime event structure S, and by $\mathcal{F}(S)$ the set of finite ones. The poset of configurations defined by S is:

$$\mathcal{D}(S) =_{\mathrm{def}} (\mathcal{F}^\infty(S), \subseteq)$$

We can now state the announced representation result ([15]):

THEOREM (FIRST REPRESENTATION THEOREM). *For any prime event structure S the poset $\mathcal{D}(S)$ is a domain, and any domain $(D, \leq)$ is isomorphic to the poset of configurations of a prime event structure. More specifically we have $(D, \leq) = \mathcal{D}(\mathcal{E}(D, \leq))$.*

For a proof, see [15] – in fact we shall prove below a slightly more general result.

2.2 Flow Event Structures.

In this paper we regard a domain as the ordered set of computations of some process; from this point of view an event – or a complete prime – may be interpreted as an elementary, i.e. irreducible step of computation, while the ordering represents causality: $e < e'$ means that the event e must occur before e' may occur. Then for the purpose of interpreting the operational semantics of programming constructs, prime event structures turn out to provide too "rigid" a notion. A discussion of this point may be found in [21]. For instance we would like to take into account the fact that a given event may be caused by two incompatible events: in other words, we would like to interpret $(a+b);c$ (where $+$ is non-deterministic choice and ; is the sequential composition) without introducing two distinct events for c. Similarly in $(\mathrm{rec}\, x.(a \parallel x))\ ;\ b$ (where rec is the fixpoint construct and $\parallel$ is parallel composition) the event of performing b – which in fact never occurs – should have infinitely many causes. Then we need to introduce a more flexible concrete presentation of domains, ruling out the axioms of finite causes and conflict heredity; in [1] we have introduced such a generalization of prime event structure. Moreover, as we said in the introduction, we need further weakenings of the notion of prime event structures to interpret neatly the operational semantics of CCS. We therefore introduce a new kind of event structure, which we call *flow event structure*:

DEFINITION (FLOW EVENT STRUCTURES). *A flow event structure is a structure $S = (E, \#, \prec)$ where*

- *E is the denumerable set of events,*
- *$\# \subseteq E \times E$ is the symmetric conflict relation,*
- *$\prec \subseteq E \times E$ is an irreflexive relation, the flow relation.*

It should be clear that any prime event structure is a flow event structure. On the other hand, in the definition of flow event structure we do not require the flow relation to be transitive nor acyclic. Note also that the conflict relation is not assumed to be irreflexive: this means that we allow *self-conflicting*, or *inconsistent* events, that is events $e \in E$ such that $e \# e$. We will see that such events cannot in general be removed from a flow event structure without affecting its set of configurations.

The first order flow event structures allow a graphical representation, with two kinds of arcs between events. In this representation we shall draw $e \prec e'$ as $e \rightarrow\!\!\!\vdash e'$. For instance

$$\mathbf{N} \qquad \begin{array}{ccccc} a & & & & \\ \downarrow & \searrow & & & \\ b & \cdots\#\cdots & c & \cdots\#\cdots & d \\ & & & \searrow & \downarrow \\ & & & & e \end{array}$$

is a flow event structure where $a \prec b$, $b \# c$, and so on. This structure does not satisfy the conflict heredity property since $c \prec e$, $c \# d$ and $d \prec e$. Note also that in $\mathbf{N}$ the flow relation is not transitive since $a \prec e$ does not hold. In a prime event structure, the conflict and flow relations are disjoint;

this is not necessarily the case in a flow event structure. For instance

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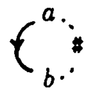

is a structure where $a \prec b$ and $a \# b$. In the following structure

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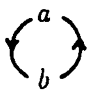
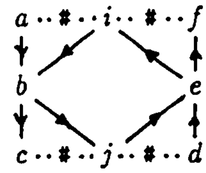


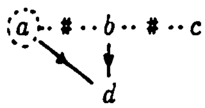
the flow is not acyclic, that is the reflexive and transitive closure of $\prec$ is not an ordering. This is also the case for

H



We shall draw self-conflicts as dotted circles around an event. For instance

T



is a flow event structure where a is self-conflicting.

We aim now at showing that flow event structures provide another concrete presentation of domains. To this end we have to define the configurations of a structure $S = (E, \#, \prec)$. Since flow event structures are not very constrained, the definition of configurations is slightly more elaborate than for prime event structures. As we said a configuration represents a computation of the system modelled by S. Since choices are resolved while computing, a configuration will be a conflict-free subset X of E. Moreover we assume a kind of "causal consistency" property, stating that an event cannot be a strict cause of itself. This property is obviously true in configurations of prime event structures, where there is a global notion of causality, namely $e' < e$ which means "e' is a cause of e". Instead we have the relation $e' \prec e$, which one can interpret as "e' is a condition for e", or "e' is a possible (immediate) cause of e". Then the causal consistency of configurations is expressed by requiring that in a configuration the relation $\prec$ generates a partial order. This (local) *causality* ordering is, denoting as usual by R^* the reflexive and transitive closure of the relation R:

$$\leq_X \;=_{\mathrm{def}}\; \prec_X^* \qquad \text{where} \qquad \prec_X = \prec \cap (X \times X)$$

This amounts to say that the relation $\prec_X$ is *acyclic*: $e \prec_X^+ e' \;\Rightarrow\; e \neq e'$, where R^+ denotes the transitive closure of R. Finally, in order to be a configuration a set X of events must satisfy, besides the finite causes property, a "downward-closure" property, meaning that an event cannot occur unless its causes have occurred. This property, expressed in terms of $\prec$, only holds up to the resolution of conflicts. More precisely, this means that if a condition e' for an event $e \in X$ does not appear in X, i.e. $e' \prec e$ and $e' \notin X$, this is because e' is discarded by another condition of e occurring in X. Let us denote by $\sharp$ the reflexive closure of the conflict relation, that is:

$$e \,\sharp\, e' \;\Leftrightarrow_{\mathrm{def}}\; e = e' \text{ or } e \# e'$$

We still denote by Cons_S the set of conflict-free sets of events. Then the configurations are defined as follows:

DEFINITION (CONFIGURATIONS). *Let $S = (E, \#, \prec)$ be a flow event structure. A configuration of S is a subset X of E such that:*

(i) *X is conflict-free: $X \in Con_S$,*

(ii) *X does not contain a causality cycle: the relation $\leq_X$ is an ordering,*

(iii) *for all $e \in X$ the set $\{e' \mid e' \in X \ \&\ e' \leq_X e\}$ is finite,*

(iv) *for all $e \in X$ if $e' \prec e$ then there exists $e'' \in X$ such that $e' \,\natural\, e'' \prec e$.*

It should be clear that for prime event structures this definition coincides with the previous one. Then we shall still denote by $\mathcal{F}^\infty(S)$ (resp. $\mathcal{F}(S)$) the set of configurations (resp. finite configurations) of a flow event structure S, and its domain of configurations $(\mathcal{F}^\infty(S), \subseteq)$ by $\mathcal{D}(S)$. Let us see some examples: the structure $\mathbf{O}$ above has just one configuration, namely $\emptyset$, while the poset of configurations of $\mathbf{O}'$ is

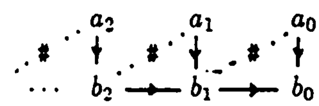
$$\begin{array}{c}\{a\}\\ |\\ \emptyset\end{array}$$

The structure $\mathbf{N}$ has a configuration $X = \{a, c, e\}$ where $a \leq_X e$, while in the configuration $Y = \{a, d, e\}$ the events a and e are not causally related. This example shows that in a flow event structure there is no global causality relation – as it is a case for a prime event structure. In the structure $\mathbf{H}$ the two events i and j are in a flow cycle ($i \prec^* j \prec^* i$), but they may occur in different configurations, namely $\{d, e, i\}$ and $\{a, b, j\}$. The poset of configurations of $\mathbf{T}$ is:

$$\begin{tikzcd}[column sep=small]
\{b,d\} \arrow[d, dash] & & \\
\{b\} \arrow[dr, dash] & & \{c\} \arrow[dl, dash] \\
& \emptyset &
\end{tikzcd}$$

Obviously a self-conflicting event cannot occur in a configuration. Let us see a last example, showing that it may be the case that, although $\{e' \mid e' \prec^* e\}$ is infinite, the event e may occur in a configuration: let

$\mathbf{U}$



In this structure the set $\{e \mid e \prec^* b_i\}$ is infinite for all i, but for instance the set $\{a_{2n}, b_{2n} \mid n \geq 0\}$ is a configuration.

It is in general not so easy to determine whether a given set of events is a configuration of a flow event structure. However we may give an alternative characterization of configurations, which is easier to manage. This characterization formalizes the idea that a computation – i.e. a configuration – may be realized as a sequence of events respecting the causality ordering. Let us introduce the notion of proving sequence with respect to a flow event structure:

DEFINITION (PROVING SEQUENCES). *Given $S = (E, \#, \prec)$, a proving sequence in S is a (finite or infinite) sequence $e_1, \ldots, e_n, \ldots$ of distinct non-conflicting events (i.e. $i \neq j \Rightarrow e_i \neq e_j$ and $\neg(e_i \# e_j)$ for all i, j) satisfying:*

$$\forall i \forall e.\ e \prec e_i \ \Rightarrow\ \exists j < i \quad e \,\natural\, e_j \prec e_i$$

Remark that any prefix of a proving sequence is a proving sequence. Given $X \subseteq E$, we shall say that a proving sequence $e_1, \ldots, e_n$ is a proof of e in X if $e = e_n$ and $\{e_1, \ldots, e_n\} \subseteq X$. Then we have:

PROPOSITION 2.3. *Given a flow event structure $S = (E, \#, \prec)$, a subset X of E is a configuration of S if and only if*

(i) X is conflict-free, and

(ii) every event $e \in X$ has a proof in X (w.r.t. S).

More precisely, X is a configuration of S if and only if X can be enumerated as $\{e_1, \ldots, e_n, \ldots\}$, where $e_1, \ldots, e_n, \ldots$ is a proving sequence.

Before proving this proposition, let us remark the following: let X be a configuration of S; recall that X, as a subset of E, is denumerable, hence $X = \{x_1, \ldots, x_n, \ldots\}$. Then an obvious idea for enumerating X as a proving sequence $e_1, \ldots, e_k, \ldots$ would be to take at each step the first x_i among minimal events, with respect to the causality ordering of X, which has not yet been enumerated. But this is not sufficient to exhaust X, as shown by the previous example: $\{a_{2n}, b_{2n} \mid n \geq 0\}$ is a configuration of U which cannot be enumerated in this way. Therefore we have to devise a more refined technique.

To prove the proposition we first establish some intermediary facts. Let us show that a subset X of a configuration Y is a configuration if and only if it is a cone of Y, with respect to the causality ordering $\leq_Y$ in Y:

LEMMA (THE STABILITY LEMMA) 2.4. *Let X and Y be two configurations of S. If $X \subseteq Y$ then*

$$e \in X \ \& \ e' \leq_Y e \ \Rightarrow \ e' \in X$$

Conversely if Y is a configuration of S and X is a cone of Y (w.r.t. $\leq_Y$) then $X \in \mathcal{F}^\infty(S)$.

PROOF: to prove the first point it is enough to show

$$e \in X \ \& \ e' \prec_Y e \ \Rightarrow \ e' \in X$$

Assume the contrary, that is $e' \notin X$: then there should exist $e'' \in X$ such that $e' \# e'' \prec e$, since X is a configuration; but this would imply $Y \notin Con_S$, for $\{e', e''\} \subseteq Y$.

Conversely if X is a cone of Y, then X clearly satisfies the points (i)-(iii) of the definition of configurations. If $e \in X$ and $e' \prec e$ is such that $e' \notin X$, then we cannot have $e' \in Y$ since X is a cone of Y. Therefore there exists $e'' \in Y$ such that $e' \# e'' \prec e$, and $e'' \in X$ since X is a cone of Y. This shows that X satisfies (iv) $\square$

An obvious consequence of this lemma is:

COROLLARY 2.5. *Let S be a flow event structure and X a configuration of S. For $e \in X$ let*

$$\lceil e \rceil_X =_{\mathrm{def}} \{e' \mid e' \in X \ \& \ e' \leq_X e\}$$

Then $\lceil e \rceil_X$ is a (finite) configuration of S.

Recall that for $X, Y \in \mathcal{F}^\infty(S)$ the compatibility relation $X \uparrow Y$ means $\exists Z \in \mathcal{F}^\infty(S)\ X \subseteq Z \ \& \ Y \subseteq Z$.

COROLLARY 2.6. *If $X, Y \in \mathcal{F}^\infty(S)$ then*

$$X \uparrow Y \ \& \ e \in X \cap Y \ \Rightarrow \ \lceil e \rceil_X = \lceil e \rceil_Y$$

PROOF: let Z be a configuration of S such that $X \subseteq Z$ and $Y \subseteq Z$. Let us show that $\lceil e \rceil_X = \lceil e \rceil_Z$ for $e \in X \cap Y$. By definition of the causality ordering $\leq_X$ we have $e' \leq_X e \Rightarrow e' \leq_Z e$, therefore $\lceil e \rceil_X \subseteq \lceil e \rceil_Z$. Conversely, we know by the previous corollary that $\lceil e \rceil_X$ is a configuration, and $\lceil e \rceil_X \subseteq Z$. Then by the stability lemma $e' \leq_Z e \Rightarrow e' \in \lceil e \rceil_X$, that is $\lceil e \rceil_Z \subseteq \lceil e \rceil_X$ Similarly $\lceil e \rceil_Y = \lceil e \rceil_Z$, hence $\lceil e \rceil_X = \lceil e \rceil_Y$ □

Now we show that two compatible configurations have a least upper bound, which is their union:

COROLLARY 2.7. *Let $X, Y \in \mathcal{F}^\infty(S)$ be two configurations such that $X \uparrow Y$. Then $X \cup Y \in \mathcal{F}^\infty(S)$.*

PROOF: let $Z \in \mathcal{F}^\infty(S)$ be such that $X \subseteq Z$ and $Y \subseteq Z$. Then by the previous lemma X and Y are two cones of Z, and clearly their union $X \cup Y$ is a cone of Z, hence a configuration □

The last step towards the proposition is a separation lemma:

LEMMA 2.8. *Let S be a flow event structure and X, Y be two configurations of S such that $X \subset Y$. Then there exists $e \in Y - X$ such that $X \cup \{e\} \in \mathcal{F}^\infty(S)$.*

PROOF: let $e \in Y - X$. Then by corollary 2.5 $\lceil e \rceil_Y$ is a configuration and $\lceil e \rceil_Y \uparrow X$, hence by the previous corollary $Z = \lceil e \rceil_Y \cup X$ is a configuration of S; moreover $Z - X$ is a finite non-empty set of events, since $\lceil e \rceil_Y$ is finite and $e \notin X$. We proceed by induction on the cardinal $k = \#(Z - X)$. If $k = 1$ then $Z - X = \{e\}$ and we are done. If $k > 1$, let $e' \in Z - X$ such that $e' \neq e$. We have $\lceil e' \rceil_Y \subset \lceil e \rceil_Y$ (by corollary 2.6, since $\lceil e \rceil_Y \uparrow Y \Rightarrow \lceil e' \rceil_{\lceil e \rceil_Y} = \lceil e' \rceil_Y$) and $\lceil e' \rceil_Y \uparrow X$. Then we use the induction hypothesis for $Z' = \lceil e' \rceil_Y \cup X$ □

We can now prove the proposition 2.3:

PROOF of the PROPOSITION: let $X = \{e_1, \ldots, e_n, \ldots\}$ where $e_1, \ldots, e_n, \ldots$ is a proving sequence. Then X is a conflict-free set of events such that every $e \in X$ has a proof in X. Let k_e be the minimal length of a proof of e in X. We show that for $e' \in X$ such that $e' \prec e$ we have $k_{e'} < k_e$. Let $e_1, \ldots, e_n$ be a proof of e in X such that $n = k_e$; then there exists $i < n$ such that $e' = e_i$, otherwise there should exist $j < n$ such that $e' \# e_j$, and this would imply $X \notin Con_S$. Since $e_1, \ldots, e_i$ is a proof of e' in X we have $k_{e'} < k_e$. This shows that $\leq_X$ is an ordering. Moreover given a proof $e_1, \ldots, e_n$ of e in X, it is easy to see by transitivity that $e' \leq_X e \Rightarrow \exists i \leq n.\ e' = e_i$, since $e_1, \ldots, e_j$ is also a proof of e_j in X for all $j \leq n$. Therefore $\{e' \mid e' \leq_X e\}$ is finite. Finally if $e' \prec e$ and $e' \notin X$ (with $e \in X$) then for any proof $e_1, \ldots, e_n$ of e in X there exists $i < n$ such that $e' \# e_i \prec e$, by definition of the notion of proof since $\{e_1, \ldots, e_n\} \subseteq X$ and $e' \notin X$. This shows $X \in \mathcal{F}^\infty(S)$.

Conversely, to show that a configuration X is a conflict-free set of events such that every $e \in X$ has a proof in X, it is enough to prove that X can be enumerated as $\{e_1, \ldots, e_n, \ldots\}$, where the sequence $e_1, \ldots, e_n, \ldots$ is a proving sequence. This is trivial if X is empty. Otherwise since X is a subset of the denumerable set of events, it may be enumerated as $\{x_1, \ldots, x_n, \ldots\}$, in such a way that if $i < j$ and $x_i = x_j$ then $x_m = x_i$ for all $m > i$. Let us define the sequence $(X_n)_{n \geq 1}$ of subsets of X as follows:

$$\begin{cases} X_1 = \lceil x_1 \rceil_X \\ X_{n+1} = X_n \cup \lceil x_{n+1} \rceil_X \quad (\text{for } n > 1) \end{cases}$$

We obviously have $X_n \subseteq X_{n+1}$ and $X = \bigcup\{X_n \mid 1 \leq n\}$. Moreover a simple induction on n shows that every X_n is a finite configuration of S:

- this is true for $n = 1$ by corollary 2.5.
- if $X_n \in \mathcal{F}(S)$, we have $X_n \uparrow \lceil x_{n+1} \rceil_X$ since these two configurations are included into X, hence

X_{n+1} is a (finite) configuration by the corollary 2.7.

By the separation lemma 2.8, for any pair of configurations Y, Z such that $Y \subset Z \subseteq X$ there exists $\varepsilon(Y,Z) \in Z - Y$ such that $Y \cup \{\varepsilon(Y,Z)\}$ is a configuration. We can now define the sequence $e_1, \ldots, e_n, \ldots$ as follows, where we let $\overline{X}_n = \{e_1, \ldots, e_n\}$:

$$\begin{cases} e_1 = \varepsilon(\emptyset, X_1) \\ e_{n+1} = \begin{cases} \varepsilon(\overline{X}_n, X_k) & \text{if } \{m \mid \overline{X}_n \subset X_m\} \neq \emptyset \text{ and } k = \inf\{m \mid \overline{X}_n \subset X_m\} \\ e_n & \text{otherwise} \end{cases} \end{cases}$$

The first point is to check that this sequence is well-defined, i.e. to check that for all n the set $\overline{X}_n$ is a configuration. This is true since if we let $\overline{X}_0 = \emptyset$ then for all $n \geq 0$ either $\overline{X}_{n+1} = \overline{X}_n \cup \{\varepsilon(\overline{X}_n, Z)\}$ for some configuration Z such that $\overline{X}_n \subset Z \subseteq X$, or $\overline{X}_{n+1} = \overline{X}_n$.

Now we prove that $X = \{e_1, \ldots, e_n, \ldots\}$. By definition $e_n \in X$ for all n. Then it is enough to show that for all n there exists m such that $X_n \subseteq \overline{X}_m$. Let us first observe that for all i such that $\overline{X}_i \subset X_n$ we have $e_{i+1} \in X_n - \overline{X}_i$, hence $\overline{X}_i \subset \overline{X}_{i+1} \subseteq X_n$. Therefore the set $\{i \mid \overline{X}_i \subset X_n\}$ is finite since X_n is finite. If there is no i such that $\overline{X}_i \subset X_n$ then we are done since in this case $X_n = \overline{X}_1$ (for $\overline{X}_1 \subseteq X_n$). Otherwise let k be the greatest integer such that $\overline{X}_k \subset X_n$. Then $\{m \mid \overline{X}_k \subset X_m\} \neq \emptyset$, and if $j = \inf\{m \mid \overline{X}_k \subset X_m\}$ then we have $j \leq n$, hence $e_{k+1} \in X_n - \overline{X}_k$, therefore $\overline{X}_k \subset \overline{X}_{k+1} \subseteq X_n$. Because of our choice for k we have $X_n = \overline{X}_{k+1}$.

Finally we prove that $e_1, \ldots, e_n, \ldots$ is a proving sequence: let $e \in E$ be such that $e \prec e_k$; since $e_k \in \overline{X}_k$ we have either $e \in \overline{X}_k$, in which case there exists $i \leq k$ such that $e = e_i$, and in fact $i < k$ since $\prec$ is irreflexive, or (since $\overline{X}_k$ is a configuration) there exists $e' \in \overline{X}_k$ such that $e \# e' \prec e_k$. In that case we have $e' = e_i$ for some $i < k$, since $\prec$ is irreflexive. This shows that $e_1, \ldots, e_n, \ldots$ is a proving sequence □

2.3 The Representation Theorem.

In this section we prove the representation theorem relating domains and flow event structures, and discuss some of its consequences. As a first step, let us relate consistency in the poset $(\mathcal{F}^\infty(S), \subseteq)$ of configurations of S with conflict-freeness in the flow event structure S – recall that for $\mathcal{X} \subseteq \mathcal{F}^\infty(S)$ the pairwise consistency predicate $\mathcal{X}\Uparrow$ means

$$\forall X, Y \in \mathcal{X}\ \exists Z \in \mathcal{F}^\infty(S)\ X \subseteq Z \ \&\ Y \subseteq Z$$

LEMMA 2.9. *Let $\mathcal{X} \subseteq \mathcal{F}^\infty(S)$ be such that $\mathcal{X}\Uparrow$. Then $\bigcup\{X \mid X \in \mathcal{X}\}$ is a configuration of S, which is the least upper bound of $\mathcal{X}$ in $\mathcal{D}(S)$. Moreover $\mathcal{X}\Uparrow$ if and only if $\bigcup\{X \mid X \in \mathcal{X}\} \in Cons_S$.*

PROOF: if $e, e' \in \bigcup\{X \mid X \in \mathcal{X}\}$ then there exist $X, Y \in \mathcal{X}$ such that $e \in X$ and $e' \in Y$. Since $\mathcal{X}\Uparrow$ there is a configuration Z such that $X \subseteq Z$ and $Y \subseteq Z$. Therefore $\neg(e \# e')$ since Z is conflict-free, hence $\bigcup\{X \mid X \in \mathcal{X}\} \in Cons_S$. Moreover if $e \in \bigcup\{X \mid X \in \mathcal{X}\}$, then $e \in X$ for some $X \in \mathcal{X}$, hence e has a proof in X which is also a proof of e in $\bigcup\{X \mid X \in \mathcal{X}\}$. Then $\bigcup\{X \mid X \in \mathcal{X}\}$ is a configuration of S, and it is obvious that this is the least upper bound of $\mathcal{X}$. The last point to note is that if $\bigcup\{X \mid X \in \mathcal{X}\} \in Cons_S$ then $\mathcal{X}\Uparrow$, since $\bigcup\{X \mid X \in \mathcal{X}\}$ is a configuration □

THEOREM (SECOND REPRESENTATION THEOREM). *For any flow event structure S the poset $\mathcal{D}(S) = (\mathcal{F}^\infty(S), \subseteq)$ is a domain. Its complete primes are the configurations $\lceil e \rceil_X$ for $X \in \mathcal{F}^\infty(S)$ and $e \in X$, and its finite points are the finite configurations $X \in \mathcal{F}(S)$. Conversely if $(D, \leq)$ is a domain then $(D, \leq)$ is isomorphic to the poset $(\mathcal{F}^\infty(S), \subseteq)$ of configurations of a flow event structure.*

PROOF: the first point to note is that $\mathcal{D}(S)$ is coherent, as shown by the previous lemma.

Let us show that for each configuration X of S and each $e \in X$ the finite configuration $\lceil e \rceil_X$ is a complete prime: if $\lceil e \rceil_X \subseteq \bigsqcup \mathcal{X}$ where $\mathcal{X}$ is a set of configurations, then $e \in Y$ for some $Y \in \mathcal{X}$. We have $Y \uparrow \lceil e \rceil_X$ and by the corollary 2.6 this implies $\lceil e \rceil_Y = \lceil e \rceil_{\lceil e \rceil_X} = \lceil e \rceil_X$, hence $\lceil e \rceil_X \subseteq Y$, so $\lceil e \rceil_X$ is a complete prime.

Since for any configuration X of S we obviously have $X = \bigsqcup\{\lceil e \rceil_X \mid e \in X\}$, is it clear that if X is complete prime then there exists $e \in X$ such that $X = \lceil e \rceil_X$. This shows that the complete primes of $\mathcal{D}(S)$ are the configurations of the form $\lceil e \rceil_X$. Moreover, the formula $X = \bigsqcup\{\lceil e \rceil_X \mid e \in X\}$ also shows that any configuration is the lub of the complete primes it dominates. Clearly the set of complete primes of $\mathcal{D}(S)$ is denumerable since a complete prime is a finite subset of the denumerable set of events. To sum up, we have shown that $\mathcal{D}(S)$ is ω-prime algebraic.

This poset is clearly finitary since each $\lceil e \rceil_X$ is a finite set of events. If $X \in \mathcal{F}^\infty(S)$ is finite, as a set of events, then it includes only finitely many configurations. Conversely for $X \in \mathcal{F}^\infty(S)$ if the set $\{Y \mid Y \in \mathcal{F}^\infty(S)\ \&\ Y \subseteq X\}$ is finite, then X is a finite set of events since the set $\{Y \mid Y \in \mathcal{F}^\infty(S)\ \&\ Y \subseteq X\}$ contains the configurations $\lceil e \rceil_X$ for all $e \in X$. Then we have proved:

$$F(\mathcal{F}^\infty(S), \subseteq) = \mathcal{F}(S)$$

Given a domain $(D, \leq)$, we already defined a prime event structure $\mathcal{E}(D, \leq)$ associated with the domain. We let the reader check that from the definition of this structure we have:

$$X \in \mathcal{F}^\infty(\mathcal{E}(D, \leq)) \Leftrightarrow \exists x \in D \quad X = \hat{x} = \{y \mid y \in Pr(D, \leq)\ \&\ y \leq x\}$$

since $(D, \leq)$ is coherent. Then $\mathcal{D}(\mathcal{E}(D, \leq)) = (D, \leq)$ since $(D, \leq)$ is ω-prime algebraic (*cf.* [20,21] for a complete proof) $\Box$

We shall regard flow event structures as semantically, or abstractly *equivalent* if their domains of configurations are isomorphic:

$$S \cong S' \Leftrightarrow_{\text{def}} \mathcal{D}(S) = \mathcal{D}(S')$$

Then a trivial consequence of the representation theorem, and of corollary 2.2, is:

COROLLARY 2.10. *Two flow event structures S and S' are equivalent if and only if their posets of finite configurations are isomorphic:*

$$S \cong S' \Leftrightarrow (\mathcal{F}(S), \subseteq) = (\mathcal{F}(S'), \subseteq)$$

Another consequence is that for any flow event structure S there exists a prime event structure S' such that $S \cong S'$. As a matter of fact, the representation theorem shows the following equivalences:

$$\mathcal{E}(\mathcal{D}(S)) \cong S \quad \text{for any flow event structure } S$$

$$\mathcal{D}(\mathcal{E}(D, \leq)) = (D, \leq) \quad \text{for all domain } (D, \leq)$$

However we will be more interested here in a stronger notion of equivalence, namely "to determine the same sets of configurations", that is:

$$S \equiv S' \Leftrightarrow_{\text{def}} \mathcal{F}^\infty(S) = \mathcal{F}^\infty(S')$$

Since $X = \bigcup\{\lceil e \rceil_X \mid e \in X\}$ for any configuration X of S we have in fact:

$$S \equiv S' \Leftrightarrow \mathcal{F}(S) = \mathcal{F}(S')$$

When $S \equiv S'$ we shall say that S and S' are *strongly*, or *concretely* equivalent. From an operational point of view this notion of equivalence seems to be the right one to use: we think of a domain of configurations as representing the computations of some given distributed system; in such an interpretation the names of events, which are occurrences of elementary actions, are meaningful – for instance one can see that in the interpretation of CCS given in [5], the events are closely related to the syntactic structure of terms. Then if one defines a translation from a given model of distributed computations into another one which "preserves the events", one gets a strong notion of "implementation": a distributed system is thus translated into another one which behaves exactly in the same way. In the next section we shall establish such a concrete relationship between flow event structures and a particular class of Petri nets. Note that the transformation $\mathcal{E} \circ \mathcal{D}$ does not "preserve the events"; this means that we do not have in general $\mathcal{E}(\mathcal{D}(S)) \equiv S$. More precisely, there exists a flow event structure whose family of configurations cannot be the one of a prime event structure:

EXAMPLE 1 (PES $\subset$ FES). Let $S = (\{e_0, e_1, e_2\}, \#, \prec)$ be the flow event structure represented by:

$$\begin{tikzcd}
e_0 \arrow[r, dash, dotted, "\#"] \arrow[dr] & & e_1 \arrow[l, dash, dotted] \arrow[dl] \\
& e_2 &
\end{tikzcd}$$

Then $\mathcal{F}^\infty(S)$ is

$$\begin{tikzcd}
\{e_0, e_2\} \arrow[d, dash] & & \{e_1, e_2\} \arrow[d, dash] \\
\{e_0\} \arrow[dr, dash] & & \{e_1\} \arrow[dl, dash] \\
& \emptyset &
\end{tikzcd}$$

Since the configurations $\{e_0, e_2\}$ and $\{e_1, e_2\}$ of S are prime, they must be associated with two distinct events e_2' and e_2'' in any prime event structure S' such that $S' \cong S$. Therefore there is no prime event structure S' such that $S' \equiv S$. Note that $\mathcal{E}(\mathcal{D}(S))$ is:

$$\begin{tikzcd}
\{e_0\} \arrow[r, dash, dotted, "\#"] \arrow[d] & \{e_1\} \arrow[d] \\
\{e_0, e_2\} \arrow[r, dash, dotted, "\#"] & \{e_1, e_2\}
\end{tikzcd}$$

(the mapping $\lceil e \rceil_X \mapsto e$ is not injective in general).

The strong equivalence of flow event structures only requires preservation of the events actually occurring in some configuration. It also preserves the *semantical conflict* relation, meaning that two events cannot occur together in a configuration; given a flow event structure $S = (E, \#, \prec)$, this relation $\#_S$ is defined by:

$$e \,\#_S\, e' \;\Leftrightarrow_{\mathrm{def}}\; \forall X \in \mathcal{F}^\infty(S)\ \{e, e'\} \not\subseteq X$$

Obviously $e \,\#\, e' \;\Rightarrow\; e \,\#_S\, e'$. We denote by $E(S)$ the set of events which are not semantically inconsistent, that is events which occur in some configuration:

$$e \in E(S) \;\Leftrightarrow_{\mathrm{def}}\; \neg(e \,\#_S\, e) \;\Leftrightarrow\; \exists X \in \mathcal{F}^\infty(S)\ e \in X$$

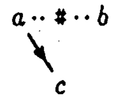
Then we have:

$$S \equiv S' \;\Rightarrow\; \begin{cases} E(S) = E(S') \quad \text{and} \\ e, e' \in E(S) \;\Rightarrow\; e \,\#_S\, e' \;\Leftrightarrow\; e \,\#_{S'}\, e' \end{cases}$$

Following Winskel, we can say that a flow event structure $S = (E, \#, \prec)$ is *full* if $E = E(S)$, and that S is *faithful* if $\#$ restricted to $E(S)$ coincides with the semantical conflict, and if every semantically inconsistent event is self-conflicting, that is:

$$\begin{cases} \forall e, e' \in E(S) \quad e \# e' \;\Leftrightarrow\; e \#_S e' \quad \text{and} \\ e \#_S e \;\Rightarrow\; e \# e \end{cases}$$

Note that we do not require a semantically inconsistent event to be conflicting with any other event – that is we do not require $\# = \#_S$. For instance the structure (called ∇' in [2]):



is not faithful. We shall see in the next section that any flow event structure is strongly equivalent to a faithful one. On the other hand, there does not in general exist a full flow event structure strongly equivalent to a given one: in particular one cannot remove the self-conflicting events without affecting the configurations. To see this point, let us prove another consequence of the stability lemma, stating that the causality orderings are in fact inherent to the family of configurations:

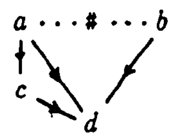
LEMMA 2.11. *For any configuration X of a flow event structure S we have*

$$e' \leq_X e \;\Leftrightarrow\; \forall Y \in \mathcal{F}^\infty(S).\; Y \subseteq X \;\&\; e \in Y \;\Rightarrow\; e' \in Y$$

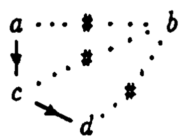
PROOF: let $e' \leq_X e$ and $Y \in \mathcal{F}^\infty(S)$ be such that $Y \subseteq X$ and $e \in Y$. Then by the corollary 2.6 we have $\lceil e \rceil_Y = \lceil e \rceil_X$, hence $e' \in Y$ since $e' \leq_X e \;\Leftrightarrow\; e' \in \lceil e \rceil_X$.

Conversely if for any configuration Y such that $Y \subseteq X$ and $e \in Y$ we have $e' \in Y$, then this holds for $Y = \lceil e \rceil_X$, hence $e' \leq_X e$ $\square$

A consequence is that for two strongly equivalent flow event structures the local causality relations are the same. Note that the covering relation associated with $\leq_X$ in a configuration X is contained in $\prec_X$, but does not necessarily coincide with this relation, as shown by the following example:

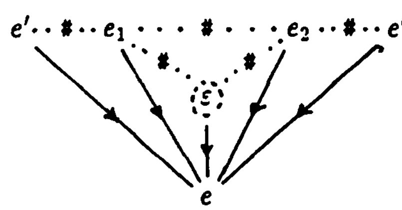


Remark that this structure is strongly equivalent to the following faithful one:

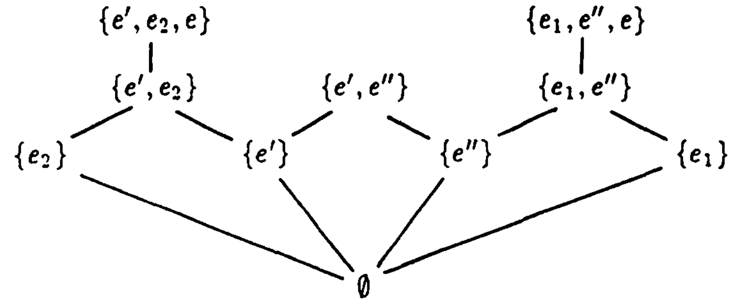


Now let us consider the following flow event structure:

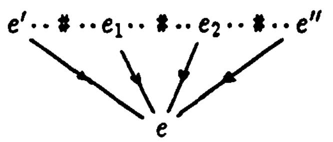
S



The domain of configurations of S is:



We let the reader convince him/herself, with the help of the previous lemma, that the only possible candidate for a full flow event structure strongly equivalent to S is:



But then $\{e', e'', e\}$ is a configuration of this structure, while it is not a configuration of S.

3. Petri Nets.

3.1 Flow Nets.

In this section we extend the result of [15], establishing the relationship between occurrence nets and prime event structures. We show a correspondence between a class of Petri nets, which is broader than the class of occurrence nets, and the class of flow event structures. The Petri nets we introduce may contain forward and backward conflicts and cycles in the flow relation, but are "semantically" acyclic. Roughly speaking this means that in any run of the net a token cannot return in a place which was previously used as a precondition.

DEFINITION (NETS). *A net is a structure $N = (B, E, \Phi, \mu_N)$ where*

- *B is the set of conditions, or places,*
- *E is the denumerable set of events, disjoint from B,*
- *$\Phi \subseteq (B \times E) \cup (E \times B)$ is the flow relation, satisfying: the set $\{b \mid \exists e, e' \ (e,b) \in \Phi \ \& \ (b,e') \in \Phi\}$ is denumerable and $\forall e \in E \ \exists b \in B.\ (b,e) \in \Phi$*
- *$\mu_N : B \to \mathbb{N}$ is the initial marking.*

A condition $b \in B$ is a *precondition* (respectively a *postcondition*) of the event $e \in E$ if $(b,e) \in \Phi$ (resp. $(e,b) \in \Phi$). Note that we require that any event has at least one precondition, but not necessarily a postcondition. The flow relation can also be represented as a mapping

$$\phi : (B \times E) \cup (E \times B) \to \{0, 1\}$$

given by $\phi(x) = 1 \Leftrightarrow x \in \Phi$. We denote by $\varphi(e, e')$ the set of places in between e and e', that is $\varphi(e, e') = \{b \mid \phi(e, b) = 1 = \phi(b, e')\}$. This set is denumerable.

A marking of the net N is any mapping $\mu : B \to \mathbb{N}$. If $\mu(b) > 0$ we shall say informally that the condition b holds at μ, or alternatively that there is a token in the place b. A marking μ *enables* an event e if all the preconditions of e hold in μ, that is $\mu(b) > 0$ for all b such that $(b,e) \in \Phi$, or more formally: $\forall b \in B.\ \phi(b,e) \leq \mu(b)$. The net N determines a labelled transition system on its markings, defined as follows:

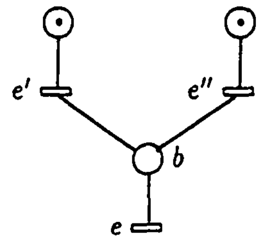
$$\mu \xrightarrow{e} \mu' \iff_{\rm def} \forall b \in B. \quad \phi(b,e) \leq \mu(b) \ \&\ \mu'(b) = \mu(b) - \phi(b,e) + \phi(e,b)$$

It should be obvious that this transition system is deterministic: if μ enables e then the next marking μ' is uniquely determined. A *firing sequence* of the net N is a finite or infinite sequence of transitions:

$$\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n \cdots$$

We can also say that the sequence $e_1, \ldots, e_n, \ldots$ of events is *firable* in the net N. Let us recall some definitions: a marking μ is reachable if there is a firing sequence such that $\mu = \mu_n$; a place $b \in B$ is called *safe* if at any reachable marking μ there is at most one token in b, that is $\mu(b) \leq 1$, and a net N is *safe* if any place of N is safe.

In this section we aim at showing that for Petri nets of a particular kind one can build a flow event structure *representing* the net in the sense that a sequence of events is firable in the net if and only if it is a proving sequence in the event structure. Conversely from a flow event structure we shall also build a net representing this structure. It should be clear that this may only hold for nets where the events of a firable sequence are distinct, what we could call *occurrence nets* – a terminology that we shall not officially introduce since it is already overloaded. Moreover the nets we are seeking should be such that one can extract from a firing sequence the causal dependencies between events. For instance we will rule out the net



In this net the sequence e', e'', e is firable, but we cannot know what token from b is used to fire e. This net corresponds to the "parallel switch" of [20], example 1.1.7, which is a typical non-stable event structure. Note that this net is not safe.

As a matter of fact, in a safe "occurrence" net one can easily define the causal, immediate dependency: e depends on e' if there is a place b between these two events, that is $b \in \varphi(e', e)$. However it will turn out that safeness is too strong a requirement for our purpose (for some account on safe nets, see Appendix B). To introduce the appropriate class of nets, which we shall call *flow nets*, we first introduce the notion of "strong postcondition" in a net N, which will play the rôle of a safe condition between two events. A place $b \in B$ is a *strong postcondition* of an event $e \in E$ if $\phi(e,b) = 1$ and in any firing sequence $\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n$ where e occurs, i.e. $e = e_j$ for some j, only e can mark b, that is:

$$\mu_0(b) + \sum_{1 \leq i \leq n} \phi(e_i, b) = 1$$

Note that if e really occurs in some firing sequence, then a strong postcondition b of e is initially unmarked, i.e. $\mu_N(b) = 0$, and b cannot be a precondition of e, otherwise b should be marked twice in the sequence. We shall use $\hat{\varphi}(e', e)$ to denote the set of strong postconditions of e' belonging to $\varphi(e', e)$.

DEFINITION (FLOW NETS). *A flow net is a net N satisfying:*

(i) *any finite firing sequence* $\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n$ *is flowing, i.e. a place cannot be used, as a precondition, more than once:* $\sum_{1 \leq i \leq n} \phi(b, e_i) \leq 1$ for all $b \in B$,

(ii) *for any firing sequence* $\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n$ *if* $\varphi(e_i, e_j) \neq \emptyset$ *then* $\widehat{\varphi}(e_i, e_j) \neq \emptyset$.

It should be clear that the *occurrence nets* of Nielsen, Plotkin and Winskel ([15,20]) are flow nets. Let us see some consequences of our definition: the first observation is that an event e cannot occur twice in a firing sequence of a flow net, since e has a precondition b, which cannot be used twice. Therefore a flow net is semantically acyclic: in a firing sequence we have $i \neq j \Rightarrow \mu_i \neq \mu_j$. Let us also remark that if two events share a precondition, i.e. $\phi(b, e) = 1 = \phi(b, e')$, then they cannot both occur in the same firing sequence. Another direct consequence of the definition is:

FACT 3.1. *If two distinct events e and e' of a flow net share a postcondition b – that is if $\phi(e, b) = 1 = \phi(e', b)$ – which is a strong postcondition of e, then they cannot both occur in the same firing sequence.*

One may remark that in the previous net $\mathbf{V}$ there is no strong postcondition for e' or e''. On the other hand, we will see that in a flow net the strong postconditions allow us to extract from any firing sequence the actual causal dependencies between the events. For flow nets (or more generally for "occurrence nets") the following definition makes sense:

DEFINITION (CONFIGURATIONS). *Given a flow net $N = (B, E, \Phi, \mu_N)$, a configuration of N is a subset X of E such that there exists a sequence*

$$\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n \cdots$$

firing the events of X, that is such that $X = \{e_1, \ldots, e_n, \ldots\}$.

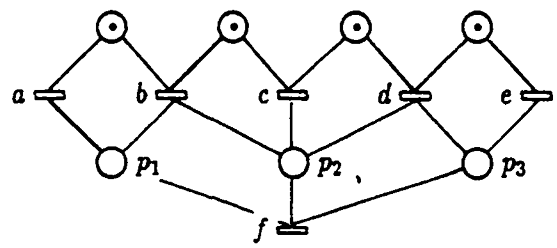
As for event structures we denote by $\mathcal{F}^\infty(N)$ the set of configurations of N; $\mathcal{F}(N)$ denotes the set of finite configurations and $\mathcal{D}(N)$ is $(\mathcal{F}^\infty(N), \subseteq)$. We also denote $N \cong N'$ the fact that these two nets are equivalent, that is $\mathcal{D}(N) \rightleftharpoons \mathcal{D}(N')$, and we still use $N \equiv N'$ to mean that N and N' are strongly equivalent, that is $\mathcal{F}^\infty(N) = \mathcal{F}^\infty(N')$. The main purpose of this paper is to show the following *representation theorem:*

THEOREM. *For any flow net N there exists a flow event structure S such that $\mathcal{F}^\infty(S) = \mathcal{F}^\infty(N)$. Conversely for any flow event structure S there exists a flow net having the same configurations.*

In fact we shall more precisely relate the notion of proving sequence in a flow event structure with the notion of firing sequence in a flow net.

Let us see an example, showing that a flow net is not necessarily strongly equivalent to a safe flow net. It is not difficult to see that the following net is a flow net:

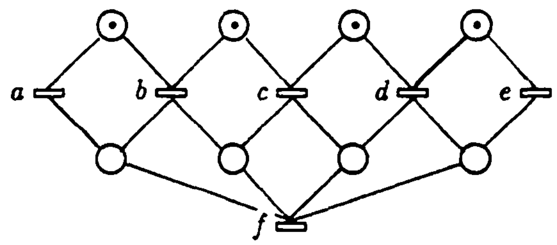
W



In this net p_1 is a strong postcondition for a and b, p_2 is a strong postcondition for c and p_3 is a strong postcondition for d and e. Note that in the firing sequence

$$\mu_0 \xrightarrow{b} \mu_1 \xrightarrow{d} \mu_2 \xrightarrow{f} \mu_3$$

we have $\mu_2(p_2) = 2$, hence this net is not safe. Remark that the safe flow net



is not equivalent (in the sense of having the same firing sequences) to the previous one: for instance a, d, f and b, e, f are not firable in this net. We shall see in the Appendix B that there is no safe flow net strongly equivalent to $\mathbf{W}$. One may characterize safe flow nets as follows:

LEMMA 3.2. *A net N is a safe flow net if and only if for any condition b and for any finite firing sequence $\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n$ the following is satisfied: $\mu_0(b) + \sum_{1 \leq i \leq n} \phi(e_i, b) \leq 1$.*

PROOF: let us first remark that if the net N satisfies the property stated in the lemma, then any postcondition in N is a strong postcondition. Moreover such a net is safe since for any firing sequence $\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n$ and for any b the i^{th} marking on b is given by:

$$(*) \qquad \mu_i(b) = \mu_0(b) - \sum_{1 \leq j \leq i} \phi(b, e_j) + \sum_{1 \leq j \leq i} \phi(e_j, b)$$

This formula also gives us $\sum_{1 \leq i \leq n} \phi(b, e_i) = \mu_0(b) - \mu_n(b) + \sum_{1 \leq i \leq n} \phi(e_i, b)$, hence the sequence is flowing.

Conversely let N be a safe flow net; the formula $(*)$ can be written, for $i = n$:

$$\mu_0(b) + \sum_{1 \leq j \leq n} \phi(e_j, b) = \mu_n(b) + \sum_{1 \leq j \leq n} \phi(b, e_j)$$

Since N is safe we have $\mu_n(b) \leq 1$, and $\sum_{1 \leq j \leq n} \phi(b, e_j) \leq 1$ since N is a flow net. Therefore to show that N satisfies the required property it is enough to prove that $\mu_n(b) = 1 = \sum_{1 \leq j \leq n} \phi(b, e_j)$ is impossible. Assume the contrary, that is $\mu_n(b) = 1$ and $\phi(b, e_j) = 1$ for some j. Since N is safe we have $\mu_{j-1}(b) \leq 1$, but μ_{j-1} enables e_j, hence $\mu_{j-1}(b) = 1$. Since $\mu_n(b) > 0$ there exists $i \geq j$ such that $\phi(e_i, b) = 1$; therefore $\varphi(e_i, e_j) \neq \emptyset$, and this implies that there is a place b' which is a strong postcondition for e_i and a precondition for e_j. Then we have $\mu_0(b') + \sum_{1 \leq l < j} \phi(e_l, b') = 0$, hence $\mu_{j-1}(b') = 0$, which contradicts the fact that μ_{j-1} enables e_j ◻

3.2 From Flow Nets to Flow Event Structures.

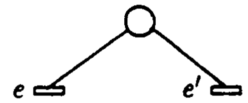
For any flow net $N = (B, E, \Phi, \mu_N)$ we shall define a flow event structure $\mathcal{E}(N)$ whose configurations are exactly the sets of events firable in sequence in N. As a first step, let us build a structure

$S(N) = (E, \#_N, \prec_N)$ as follows: two events are conflicting if they cannot both occur in a firing sequence, that is

$$e \#_N e' \Leftrightarrow_{\mathrm{def}} \forall X \in \mathcal{F}^\infty(N) \quad \{e, e'\} \not\subseteq X$$

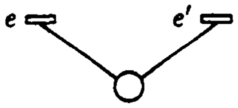
Then an event is inconsistent, that is self-conflicting, if and only if it is not firable, from any reachable marking. For instance if e has a precondition b, i.e. $\phi(b, e) = 1$, which is initially unmarked, i.e. $\mu_N(b) = 0$, and which is not a postcondition of another event, i.e. $\forall e\ \phi(e, b) = 0$, then e is inconsistent. Note that such an inconsistent event is in conflict with any other event. We have seen previously other examples of conflict, namely, using the notation $\sharp_N$ for the reflexive closure of $\#_N$:

- $\exists b.\ \phi(b, e) = 1 = \phi(b, e') \Rightarrow e \sharp_N e'$, a situation which may be drawn:



$$\Rightarrow e \sharp_N e'$$

- if $\exists b.\ \phi(e, b) = 1 = \phi(e', b)$, that is



and b is a strong postcondition of e then $e \sharp_N e'$.

The flow relation $e' \prec_N e$ of $S(N)$ means that e' and e may both occur in a firing sequence, and that there is a strong postcondition of e' which is a precondition of e:

$$e' \prec_N e \Leftrightarrow_{\mathrm{def}} \neg(e' \#_N e) \ \&\ \widehat{\varphi}(e', e) \neq \emptyset$$

Let us define the flow relation $\prec_N^b$ relative to a given place b, initially unmarked, by:

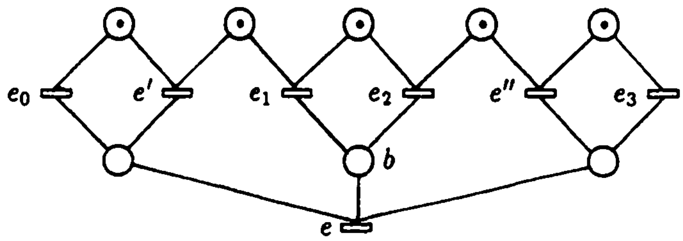
$$e' \prec_N^b e \Leftrightarrow_{\mathrm{def}} \mu_N(b) = 0 \ \&\ \neg(e' \#_N e) \ \&\ b \in \varphi(e', e)$$

Then it is easy to see that

$$e' \prec_N e \Leftrightarrow \exists b \in B.\ e' \prec_N^b e$$

Note that the relation $\prec_N$ is irreflexive, since a strong postcondition of a consistent event cannot be a precondition of this event. Therefore $S(N)$ is a flow event structure. However this structure does not quite achieve our purpose; consider for instance the safe flow net:

N



Then in $S(\mathbf{N})$ the conflict relation is the one indicated by the shared initial places, that is $e_0 \#_{\mathbf{N}} e'$, $e' \#_{\mathbf{N}} e_1$, and so on. Similarly it is easily seen that we have $e_0 \prec_{\mathbf{N}} e$, $e' \prec_{\mathbf{N}} e$, and so on. Then $\{e', e'', e\}$ is a configuration of $S(\mathbf{N})$, since for any i the event e_i, which is a possible immediate cause for e, is in conflict with e' or e''. However this should not be the case, since the condition b between e and e_1, e_2 must hold for e to be enabled, and neither e' nor e'' may fill in this place: the set $\{e', e'', e\}$ is not a configuration of $\mathbf{N}$.

The point is that a configuration $X \in \mathcal{F}^\infty(S(N))$ need not intersect all the sets $\{e' \mid e' \prec_N^b e\}$ for $e \in X$ and $b \in \varphi(e', e)$. To overcome this difficulty, we add to $S(N)$ new self-conflicting events. Let us introduce the set Ω_N of triples (X, b, e) which must be discarded, and will be the new events:

$$(X, b, e) \in \Omega_N \;\Leftrightarrow_{\text{def}}\; \begin{cases} X \in \mathcal{F}(S(N)) \;\&\; e \in X \quad \text{and} \\[2ex] \{e' \mid e' \prec_N^b e\} \neq \emptyset \quad \text{and} \\[2ex] X \cap \{e' \mid e' \prec_N^b e\} = \emptyset \end{cases}$$

Note that Ω_N is denumerable since we have, denoting $\mathit{Fin}(E)$ the set of finite subsets of E:

$$\Omega_N \subseteq \mathit{Fin}(E) \times \big(\{b \mid \exists e, e' \; \phi(e', b) = 1 = \phi(b, e)\}\big) \times E$$

Let us now define $\mathcal{E}(N) = (E_N, \#_N, \prec_N)$, where we use the same notation as above for the conflict and flow relation since they coincide on the set E of events of the net, as follows:

- $E_N = E \cup \Omega_N$ (assuming that $E \cap \Omega_N = \emptyset$); in what follows we shall denote the event (X, b, e) of Ω_N by $\varepsilon_{(X,b,e)}$;

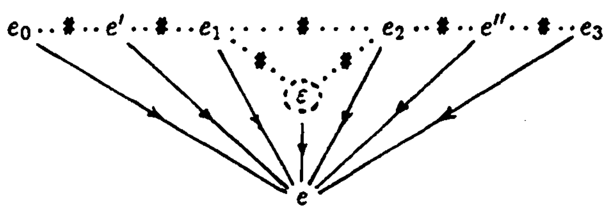
An event $\varepsilon_{(X,b,e)}$ is self-conflicting and in conflict with any event e' which causes e by means of b (that is $e' \prec_N^b e$):

- $$e \,\#_N\, e' \;\Leftrightarrow_{\text{def}}\; \begin{cases} e \in \Omega_N \;\&\; e' = e \quad \text{or} \\[2ex] \exists (X, b, e'') \in \Omega_N \quad \begin{array}{l} e = \varepsilon_{(X,b,e'')} \;\&\; e' \prec_N^b e'' \quad \text{or symmetrically} \\ e' = \varepsilon_{(X,b,e'')} \;\&\; e \prec_N^b e'' \quad \text{or} \end{array} \\[2ex] e, e' \in E \;\&\; \forall X \in \mathcal{F}^\infty(N) \quad \{e, e'\} \not\subseteq X \end{cases}$$

An event $\varepsilon_{(X,b,e)}$ is a cause for e:

- $$e' \prec_N e \;\Leftrightarrow_{\text{def}}\; \begin{cases} e, e' \in E \;\&\; \exists b \in B.\; e' \prec_N^b e \quad \text{or} \\[2ex] e' = \varepsilon_{(X,b,e)} \quad \text{for some } X \text{ and } b \end{cases}$$

It is obvious that $\mathcal{E}(N)$ is a flow event structure. We shall use the notation Con_N for $\mathit{Con}_{\mathcal{E}(N)}$. Let us see an example; the structure $\mathcal{E}(\mathbf{N})$ built from the net above can be drawn:



We have:

Lemma 3.3. *Let $X \in \mathcal{F}(\mathcal{E}(N))$ and $e \in X$. Then for all $b \in B$ such that $\{e' \mid e' \prec_N^b e\} \neq \emptyset$ we have $X \cap \{e' \mid e' \prec_N^b e\} \neq \emptyset$.*

PROOF: let us first check that $\mathcal{F}(\mathcal{E}(N)) \subseteq \mathcal{F}(S(N))$; clearly any subset of E_N which is conflict-free with respect to $\mathcal{E}(N)$ is a subset of E, and is also conflict-free with respect to $S(N)$). If X is a finite configuration of $\mathcal{E}(N)$ then it can be enumerated as in proposition 2.3, that is as a proving sequence $e_1, \ldots, e_n$. Since the restriction to E of the flow relation of $\mathcal{E}(N)$ coincides with the flow relation of $S(N)$, the sequence $e_1, \ldots, e_n$ is also a proving sequence with respect to $S(N)$.

Now let $X \in \mathcal{F}(\mathcal{E}(N))$, $e \in X$ and $b \in B$ be such that $\{e' \mid e' \prec_N^b e\} \neq \emptyset$, and assume that $X \cap \{e' \mid e' \prec_N^b e\} = \emptyset$. Since X is a configuration of $S(N)$ we have $(X, b, e) \in \Omega_N$ by definition, and since $\varepsilon_{(X,b,e)} \prec_N e$ and X is a configuration of $\mathcal{E}(N)$ we have either $\varepsilon_{(X,b,e)} \in X$, but this is impossible since $\varepsilon_{(X,b,e)}$ is inconsistent, or $\exists e'' \in X\ \varepsilon_{(X,b,e)} \#_N e'' \prec_N e$, but then either $e'' = \varepsilon_{(X,b,e)}$, and we just saw that this is impossible, or $e'' \in \{e' \mid e' \prec_N^b e\}$, contradicting the hypothesis $X \cap \{e' \mid e' \prec_N^b e\} = \emptyset$. In any case we get a contradiction, hence $X \cap \{e' \mid e' \prec_N^b e\} \neq \emptyset$ □

We can now show that the configurations of the flow event structure $\mathcal{E}(N)$ are the sets of events firable in sequence in N:

PROPOSITION 3.4. *For any flow net N we have $\mathcal{F}^\infty(\mathcal{E}(N)) = \mathcal{F}^\infty(N)$. More precisely a sequence $e_1, \ldots, e_n, \ldots$ is firable in N if and only if it is a proving sequence in $\mathcal{E}(N)$. Moreover the structure $\mathcal{E}(N)$ is faithful.*

PROOF: let X be a configuration of $\mathcal{E}(N)$, and assume that X is enumerated as in the proposition 2.3, that is as a proving sequence $e_1, \ldots, e_n, \ldots$ of $\mathcal{E}(N)$. Note that $X \subseteq E$ since $X \in Con_N$. We show by induction on n that there is a firing sequence $\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n$ (w.r.t. N) such that μ_n enables e_{n+1}:

• since the sequence e_1 is a proving sequence we have $\{e \mid e \prec_N e_1\} = \emptyset$, and e_1 is consistent since X is consistent. Let b be a precondition of e_1. Since e_1 is consistent, it occurs in a firing sequence σ, where b holds. Let us assume that $\mu_N(b) = 0$; then there should exist an event e occurring in σ such that $e \prec_N^b e_1$. But this contradicts $\{e \mid e \prec_N e_1\} = \emptyset$, therefore we have $\phi(b, e_1) \leq \mu_N(b)$.

• let us assume that $\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n$ is a firing sequence, and let $b \in B$ be a precondition of e_{n+1}. We have to show that $\mu_n(b) > 0$. Since e_{n+1} is consistent, it occurs in a firing sequence σ where b holds. Then we have $\mu_0(b) > 0$ or $e \prec_N^b e_{n+1}$ for some e occurring in σ. If $\mu_0(b) > 0$, let us assume that $\mu_n(b) = 0$; then there would exist k ($k \leq n$) such that $\phi(b, e_k) = 1$, but then $e_k \#_N e_{n+1}$, which does not hold. Therefore $\mu_0(b) > 0 \Rightarrow \mu_n(b) > 0$. If $\{e \mid e \prec_N^b e_{n+1}\} \neq \emptyset$, by the previous lemma 3.3 we know that $\{e_1, \ldots, e_{n+1}\} \cap \{e \mid e \prec_N^b e_{n+1}\} \neq \emptyset$ since $\{e_1, \ldots, e_{n+1}\}$ is a configuration of $\mathcal{E}(N)$. Then for some $i \leq n$ we have $\phi(e_i, b) = 1$, hence $\mu_i(b) > 0$, and the same argument as above shows that we cannot have $\mu_n(b) = 0$. This shows that any precondition of e_{n+1} holds at μ_n.

Conversely let $\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n \cdots$ be a firing sequence of the net N. We prove that $e_1, \ldots, e_n, \ldots$ is a proving sequence of $\mathcal{E}(N)$. By definition we have $\{e_1, \ldots, e_n, \ldots\} \in Con_N$, and $i \neq j \Rightarrow e_i \neq e_j$ since N is a flow net. Now assume that $e \prec_N e_k$. We want to show that this implies $\exists i < k.\ e \natural_N e_i \prec e_k$. There are two cases:

• if $e \in \Omega_N$ then for some Y and b we have $e = \varepsilon_{(Y,b,e_k)}$. Then $\phi(b, e_k) = 1$ and $\{e' \mid e' \prec_N^b e_k\} \neq \emptyset$, which implies $\mu_0(b) = 0$. Since we have $\mu_{k-1}(b) > 0$ there exists $i < k$ such that $\phi(e_i, b) = 1$; then it is easy to check that $e_i \prec_N^b e_k$, and by definition of $\#_N$ we have $e \#_N e_i$.

• if $e \in E$ then $e \prec_N e_k \Rightarrow \neg(e \#_N e_k)$, hence there exists a firing sequence where both e and e_k occur. By definition of $e \prec_N e_k$ there exists a strong postcondition b of e such that $b \in \varphi(e, e_k)$.

This implies $\mu_0(b) = 0$, for e is consistent. Since we have $\mu_{k-1}(b) > 0$ there exists $i < k$ such that $\phi(e_i, b) = 1$, hence by the fact 3.1 $\phi(e,b) = 1 \Rightarrow e = e_i$ or $e \,\#_N\, e_i$.

Finally it is obvious that $\mathcal{E}(N)$ is faithful since from the previous points for $e, e' \in E(\mathcal{E}(N))$:

$$\exists X \in \mathcal{F}^\infty(\mathcal{E}(N)) \quad \{e,e'\} \not\subseteq X \;\Leftrightarrow\; \exists X \in \mathcal{F}^\infty(N) \quad \{e,e'\} \not\subseteq X$$

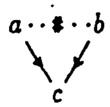
that is $e \,\#_N\, e'$ by definition; moreover if e does not occur in any configuration of $\mathcal{E}(N)$ then we have either $e \in E \;\&\; e \,\#_N\, e$ or $e \in \Omega_N$, hence $e \,\#_N\, e$ $\square$

One may also note that in the structure $\mathcal{E}(N)$ the flow relation is consistent, that is:

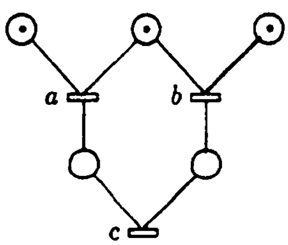
$$e \prec_N e' \Rightarrow \neg(e \,\#_N\, e')$$

3.3 From Flow Event Structures to Flow Nets.

Conversely, we may associate with a flow event structure a flow net such that the sets of events firable in sequence are the configurations of the structure. A simple construction of a net associated with a flow event structure $S = (E, \#, \prec)$ could be the following: set a marked precondition for each conflicting pair of events, an unmarked precondition for each self-conflicting event, a marked precondition for each initial event, i.e. e such that $\{e' \mid e' \prec e\} = \emptyset$, and an unmarked condition between events e and e' if $e \prec e'$. But this is incorrect if the structure does not satisfy a conflict heredity property; for instance if S is the structure:



then we would get the net:



where c is not firable. We then have to refine this simple idea. Somehow a precondition should group together conflicting causes.

Let $S = (E, \#, \prec)$ be a flow event structure. We define the net, having the same events, $\mathcal{N}(S) = (B_S, E, \Phi_S, \mu_S)$ as follows: conditions are sets of events of two types, i.e. $B_S = B'_S \cup B''_S$. The first type is used to solve conflicts; a condition of this type is either the empty set, to inhibit self-conflicting events, or a maximal pairwise conflicting set of events, what we could call a *cell*. Let us denote by Γ_S the set of pairwise conflicting sets of events:

$$F \in \Gamma_S \;\Leftrightarrow_{\mathrm{def}}\; F \subseteq E \;\&\; e,e' \in F \Rightarrow e \,\sharp\, e'$$

This set is ordered by inclusion, and we shall denote by $\bar{\Gamma}_S$ the set of maximal elements of Γ_S, that is

$$F \in \bar{\Gamma}_S \;\Leftrightarrow_{\mathrm{def}}\; F \in \Gamma_S \;\&\; (F' \in \Gamma_S \;\&\; F \subseteq F' \Rightarrow F' = F)$$

It is easily seen, using Zorn's Lemma(†), that for any $F \in \Gamma_S$ there exists $H \in \bar{\Gamma}_S$ such that $F \subseteq H$. Then a condition of the first kind, B'_S, is the empty set or an element of $\bar{\Gamma}_S$, in other words:

$$B'_S =_{\text{def}} \{\emptyset\} \cup \bar{\Gamma}_S$$

Conditions of the second kind, B''_S, are used to record the flow relation of the event structure. These are pairs (G, e) where G is the set of causes for e which are in conflict with, or equal to a given condition e' for e. Two events of G are not necessarily in conflict; therefore these places will not be safe in general. On the other hand, the place (G, e) will be a strong postcondition for e'. Let us introduce a notation for such G's:

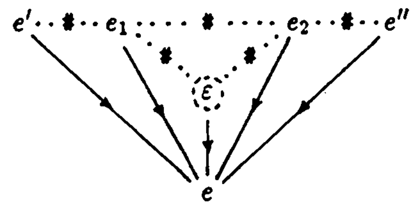
$$\kappa_S(e', e) =_{\text{def}} \{e'' \mid e' \,\sharp\, e'' \prec e\}$$

Then B''_S is given by:

$$(G, e) \in B''_S \;\Leftrightarrow_{\text{def}}\; \exists e' \prec e.\;\; G = \kappa_S(e', e)$$

It should be clear that B''_S is denumerable, since there is a surjective mapping from the subset $\prec$ of $E \times E$ onto B''_S.

For instance, with the previous example of flow event structure, besides the empty set there are two conditions of the first type, B'_S, namely $\{a, b\}$ and $\{c\}$, and only one condition of the second type, B''_S, which is $(\{a, b\}, c)$. Let us see another example: for the structure S of the previous section, that is:



Here the set B'_S consists of $\emptyset$, $b_1 = \{e', e_1\}$, $b_2 = \{e_1, \varepsilon, e_2\}$, $b_3 = \{e_2, e''\}$ and $\{e\}$, while the conditions of B''_S are (b_1, e), $(b_1 \cup b_2, e)$, (b_2, e), $(b_2 \cup b_3, e)$ and (b_3, e).

The flow relation Φ_S of $\mathcal{N}(S)$ is the least one such that:

$$e \,\sharp\, e \;\Rightarrow\; (\emptyset, e) \in \Phi_S$$

$$F \in B'_S \;\&\; e \in F \;\Rightarrow\; (F, e) \in \Phi_S$$

$$(G, e) \in B''_S \;\&\; e' \in G \;\Rightarrow\; (e', (G, e)) \in \Phi_S$$

$$(G, e) \in B''_S \;\Rightarrow\; ((G, e), e) \in \Phi_S$$

Note that a condition of B'_S cannot be a postcondition of an event. In what follows we shall use indexed notations for the mappings related to the flow relation, i.e. ϕ_S and φ_S. Finally the initial marking is given by:

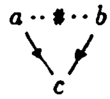
$$\mu_S(b) =_{\text{def}} \begin{cases} 1 & \text{if } b \in B'_S \;\&\; b \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

LEMMA 3.5. *For any flow event structure $S = (E, \sharp, \prec)$ the structure $\mathcal{N}(S)$ is a net.*

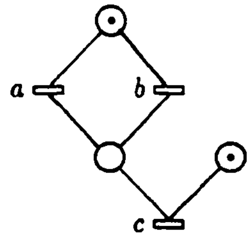
(†) Let A be a set and $\mathcal{A}$ a set of subsets of A. The family $\mathcal{A}$ is *inductive* if any chain (totally ordered subset of $\mathcal{A}$, w.r.t. inclusion) is bounded in $\mathcal{A}$. Zorn's Lemma asserts that if $\mathcal{A}$ is inductive, then every element of $\mathcal{A}$ is contained into a maximal element of $\mathcal{A}$.

PROOF: we already saw, using Zorn's Lemma, that for any $e \in E$ there exists $F \in \bar{\Gamma}_S$ such that $e \in F$, and by definition $(F, e) \in \Phi_S$. Moreover we saw that $B''_S = \{b \mid b \in B_S \ \& \ \exists e, e' \ \phi_S(e', b) = 1 = \phi_S(b, e)\}$ is denumerable ◻

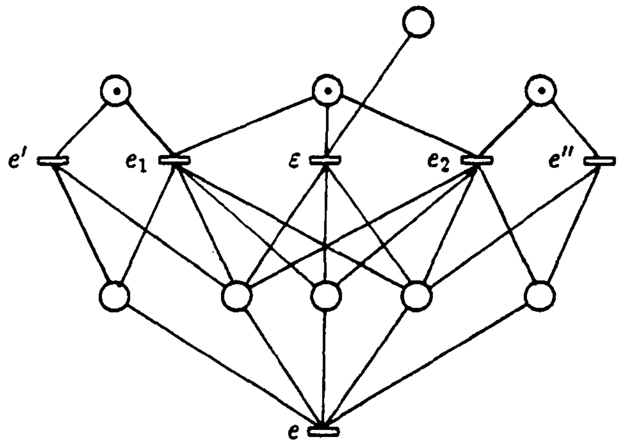
For instance from the structure



we get the net (omitting the isolated empty condition):

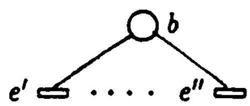


while from **S** we get (omitting the useless precondition $\{e\}$ for e):

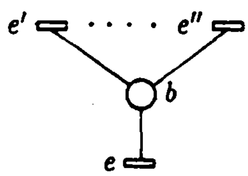


One may observe that the net $\mathcal{N}(S)$ has a "canonical form". Let us see this point in some detail. Remark that in defining flow nets we used *behavioural* properties, that is properties expressed in terms of firing sequences. On the other hand, $\mathcal{N}(S)$ has some remarkable *structural* properties, stated in terms of the "local" flow relation Φ and initial marking μ_N of a net $N = (B, E, \Phi, \mu_N)$. Roughly speaking, we can distinguish in $\mathcal{N}(S)$ two kinds of places:

• *choice places*: these places are *not* postconditions of any event; on the other hand, one such place b may be *forward branched*, that is $\{e \mid \phi(b, e) = 1\}$ may contain more than one element. In picture:



• *causal places*: these places are *both* postconditions and preconditions; one such place b is possibly *backwards branched* , that is $|\{e \mid \phi(e, b) = 1\}| \geq 1$, but is precondition of at *most one* event. In picture:



More precisely, $\mathcal{N}(S)$ satisfies the properties:

(i) a forward branched place is a choice place, that is:

$$\exists e', e'' \ (e' \neq e'' \ \& \ \phi(b,e') = 1 = \phi(b,e'')) \ \Rightarrow \ \forall e. \ \phi(e,b) = 0$$

An equivalent formulation of this property is: if a place $b \in B$ is a postcondition of some event then it is the precondition of at most one event:

$$\exists e. \ \phi(e,b) = 1 \ \Rightarrow \ (\phi(b,e') = 1 = \phi(b,e'') \ \Rightarrow \ e' = e'')$$

(ii) only choice places can be initially marked, with at most one token:

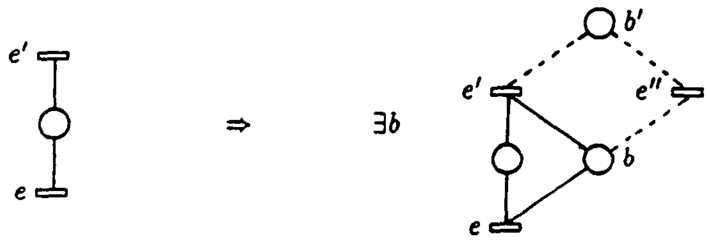
$$\mu_N(b) > 0 \ \Rightarrow \ \mu_N(b) = 1 \ \& \ \forall e. \ \phi(e,b) = 0$$

This property implies that if b is a postcondition of some event, then it is initially unmarked. The last property concerns the strong postconditions, which appear in a rather special way in $\mathcal{N}(S)$, namely:

(iii) if there is a place between e' and e then there exists one such place b such that if $\phi(e'',b) = 1$ then e' and e'' share a precondition, that is:

$$\varphi(e',e) \neq \emptyset \ \Rightarrow \ \exists b \in \varphi(e',e) \ \forall e'' \ (\phi(e'',b) = 1 \ \Rightarrow \ \exists b'. \ \phi(b',e') = 1 = \phi(b',e''))$$

This property may be drawn as



We shall say that a net is a *regular flow net* if it satifies the properties (i)-(iii) above (note: here regularity refers to the fact that the properties are structural, but has nothing to do with the fact that the finite firing sequences of events form a regular language). This terminology is justified by the following:

LEMMA 3.6. *Any regular flow net is a flow net.*

PROOF: let $\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n$ be a firing sequence. Let us first show that if $\phi(b,e_i) = 1 = \phi(b,e_j)$ for some b then $e_i = e_j$: if $e_i \neq e_j$ then b is a forward branched place, hence not a postcondition. Therefore if for instance $i < j$ we have $\mu_0(b) \geq \mu_{i-1}(b) > 0$, hence $\mu_{i-1}(b) = 1$, $\mu_i(b) = 0$ and thus $\mu_{j-1}(b) = 0$, contradicting the fact that e_j is enabled at μ_{j-1}.

Now let us show that $i < j \Rightarrow e_i \neq e_j$; assume the contrary, and let $i = \inf\{l \mid \exists k > l \ e_k = e_l\}$ There exists $b \in B$ such that $\phi(b,e_i) = 1$, and we have $\mu_{i-1}(b) > 0$. If b is not a postcondition of some event, then b must be initially marked, and then $\mu_0(b) \geq \mu_{i-1}(b)$, hence $\mu_{i-1}(b) = 1$, and thus $\mu_i(b) = 0$. But then we cannot have $\mu_{j-1}(b) > 0$, contradicting the fact that μ_{j-1} enables e_j. Therefore there exists $e \in E$ such that $\phi(e,b) = 1$. In this case we have $\mu_0(b) = 0$, hence there exists $h < i$ such that $\phi(e_h, b) = 1$ since $\mu_{i-1}(b) > 0$. We then have $\varphi(e_h, e_i) \neq \emptyset$, therefore there exists $\bar{b} \in B$ such that $\bar{b} \in \varphi(e_h, e_i)$ and $\phi(e', \bar{b}) = 1 \Rightarrow \exists b'. \ \phi(b', e) = 1 = \phi(b', e_h)$. Now for any k if $\phi(e_k, \bar{b}) = 1$ then there is a place b' such that $\phi(b', e_k) = 1 = \phi(b', e_h)$, hence $e_k = e_h$ by the previous point. By the minimality of i this implies $k = h$, but then $\mu_{j-1}(\bar{b}) = 0$ since

$$\begin{aligned}\mu_{j-1}(\bar{b}) &= \mu_0(\bar{b}) - \sum_{1 \leq l < j} \phi(\bar{b}, e_l) + \sum_{1 \leq l < j} \phi(e_l, \bar{b}) \\ &= 1 - \sum_{1 \leq l < j} \phi(\bar{b}, e_l) \leq 0\end{aligned}$$

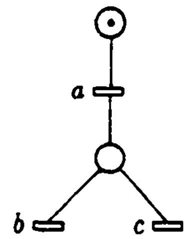
contradicting the fact that e_j is enabled at μ_{j-1}.

We have shown that if $\phi(b, e_i) = 1 = \phi(b, e_j)$ then $i = j$, hence the sequence is flowing, and if $\varphi(e_i, e_j) \neq \emptyset$ it is easy to see using the previous points that the place b such that

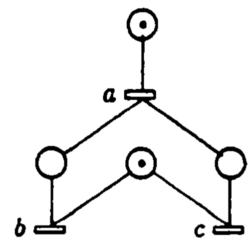
$$b \in \varphi(e_i, e_j) \ \& \ \forall e. \ \big(\phi(e, b) = 1 \Rightarrow \exists b'. \ \phi(b', e) = 1 = \phi(b', e_i)\big)$$

is a strong postcondition for e_i ❑

Let us see an example: the net



is a flow net which does not satisfies (i); but it is strongly equivalent to the regular net:



We are now ready to prove the second half of the representation theorem:

PROPOSITION 3.7. *Let $S = (E, \#, \prec)$ be a flow event structure. Then $\mathcal{N}(S)$ is a (regular) flow net such that $\mathcal{F}^\infty(\mathcal{N}(S)) = \mathcal{F}^\infty(S)$. More precisely a sequence $e_1, \ldots, e_n, \ldots$ of events is firable in $\mathcal{N}(S)$ if and only if it is a proving sequence in S.*

PROOF: from the previous lemma we know that $\mathcal{N}(S)$ is a flow net.

Now let X be a configuration of S, enumerated as in the proposition 2.3, that is as a proving sequence $e_1, \ldots, e_m, \ldots$ of S. Note that $\neg(e_i \# e_i)$ since X is conflict-free. We show by induction on n that the sequence $e_1, \ldots, e_n$ may be fired in $\mathcal{N}(S)$. It is enough to prove that (for $n \geq 0$) if $\mu_0 = \mu_S \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n$ is a firing sequence, then μ_n enables e_{n+1}:

• if $F \in B'_S$ is a precondition of e_{n+1}, then clearly we cannot have $F = \emptyset$ since e_{n+1} is consistent. Therefore $e_{n+1} \in F$; let us show that $\mu_n(F) \geq 1$: assume the contrary, that is $\mu_n(F) = 0$; then, since $\mu_S(F) = 1$, there should exist e_i (with $i < n$, hence $e_i \neq e_{n+1}$) such that $e_i \in F$. But this is impossible since we should then have $e_i \# e_{n+1}$, contradicting the fact that X is conflict-free.

• if $(G, e_{n+1}) \in B''_S$, let us show that $\mu_n(G, e_{n+1}) \geq 1$. By definition of the net $\mathcal{N}(S)$ we have $G = \kappa_S(e, e_{n+1})$ for some e such that $e \prec e_{n+1}$. Since $e_1, \ldots, e_{n+1}$ is a proving sequence, there exists $i \leq n$ such that $e_i \prec e_{n+1}$ and $e = e_i$ or $e \# e_i$. In any case $e_i \in G$, hence $\phi_S(e_i, (G, e_{n+1})) = 1$, which implies $\mu_i(G, e_{n+1}) > 0$, therefore $\mu_n(G, e_{n+1}) > 0$ since $\mathcal{N}(S)$ is a flow net.

This shows that there is a sequence of transitions in $\mathcal{N}(S)$ firing the events $e_1, \ldots, e_n, \ldots$ in that order, hence if X is a configuration of S then it is also a configuration of $\mathcal{N}(S)$.

Now let $\mu_0 = \mu_S \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n \cdots$ be a firing sequence in $\mathcal{N}(S)$. We show that $e_1, \ldots, e_n, \ldots$ is a proving sequence of S. Assume that this sequence is not conflict-free, or contains twice the same event. Then either $e_i \# e_i$ for some i, but this is impossible since $\emptyset$ is a precondition of e_i, or there would exist i and j such that $i \neq j$ and $e_i \sharp e_j$. Then there would exist a choice place b such that $\{e_i, e_j\} \subseteq b$, that is used twice in the given firing sequence, a contradiction.

Now assume that $e \prec e_k$. We have to prove that $\exists i < k.\ e \sharp e_i \prec e_k$. By definition of $\mathcal{N}(S)$ if we let $G = \kappa_S(e, e_k)$ then $(G, e_k) \in B''_S$ and $\phi_S(G, e_k) = 1$. Since μ_{k-1} enables e_k we have $\mu_{k-1}(G, e_k) > 0$, hence there exists $i < k$ such that $\phi_S(e_i, (G, e_k)) = 1$ since (G, e_k) is initially unmarked. By definition of G this implies either $e = e_i$ or $e \# e_i$.

Consequently, by the proposition 2.3, the set $X = \{e_1, \ldots, e_n, \ldots\}$ is a configuration of S, therefore any configuration of $\mathcal{N}(S)$ is a configuration of S ◻

This establishes the representation theorem relating flow nets and flow event structures. Let us see some consequences of this result – or more precisely of the constructions we used to prove it. The first one is that the poset of configurations of a flow net is a domain:

COROLLARY 3.8. *For any flow net N the poset $\mathcal{D}(N)$ is a domain.*

Another consequence is that we can now prove that any flow event structure is strongly equivalent to a faithful one:

COROLLARY 3.9. *For any flow event structure S there exists a faithful flow event structure S' such that $S' \equiv S$.*

PROOF: this is true if we let $S' = \mathcal{E}(\mathcal{N}(S))$ ◻

Recall also that in $\mathcal{E}(\mathcal{N}(S))$ the flow and conflict relations are disjoint; more precisely in this structure we have:

$$e' \prec e \ \Rightarrow\ \neg(e' \# e) \ \&\ \neg(e \# e)$$

Similarly, the transformation $\mathcal{E} \circ \mathcal{N}$ shows that we can find for any flow net a strongly equivalent regular net:

COROLLARY 3.10. *For any flow net N there exists a regular flow net N' such that $N' \equiv N$.*

PROOF: this is true if we let $N' = \mathcal{N}(\mathcal{E}(N))$ ◻

In fact we can say a little bit more: the structure $\mathcal{E}(N)$ is faithful; then it is easy to see (from the definition of $\mathcal{N}$) that in $\widehat{N} = \mathcal{N}(\mathcal{E}(N))$ we can characterize the semantical conflict as follows:

$$e \#_{\widehat{N}} e' \ \Leftrightarrow\ \begin{cases} e' = e \ \&\ \exists b.\ \mu_{\widehat{N}}(b) = 0 \ \&\ \phi(b, e) = 1 \ \&\ \forall e''\ \phi(e'', b) = 0 \quad \text{or} \\ e \neq e' \ \&\ \exists b\ \phi(b, e) = 1 = \phi(b, e') \ \&\ \forall e''\ \phi(e'', b) = 0 \end{cases}$$

In other words, the semantical conflict in $\widehat{N}$ is just the local "immediate" conflict, i.e. the direct conflict of [15].

Appendix A: Stable Event Structures.

In [21] Winskel studies another concrete presentation of domains, namely the (second order) stable event structures $I = (E, \#, \vdash)$. Let us recall the definition, with some slight modifications with respect to Winskel's one, where we still denote Con_I the set of conflict-free sets of events:

DEFINITION (STABLE EVENT STRUCTURES). *A stable event structure is a structure* $I = (E, \#, \vdash)$ *where*

- E *is the denumerable set of events,*
- $\# \subseteq E \times E$ *is a symmetric relation, the conflict relation.*
- $\vdash \subseteq P(E) \times E$ *is the enabling relation, satisfying:*

(i) *consistency:* $F \vdash e \Rightarrow F \cup \{e\} \in Con_I$

(ii) *stability:* $F \vdash e \;\&\; G \vdash e \;\&\; F \cup G \in Con_I \Rightarrow F = G$

In fact this definition involves the *minimal enabling* relation (*cf.* [20,21]), and this induces a slight modification in the definition of the configurations. Let us define a *proving sequence* (w.r.t. I) as a finite conflict-free sequence $e_1, \ldots, e_n$ of distinct events satisfying:

$$\forall i \leq n \; \exists F \subseteq \{e_1, \ldots, e_{i-1}\} \quad F \vdash e_i$$

As before, we say that such a sequence is a proof of e in X if $e_n = e$ and $\{e_1, \ldots, e_n\} \subseteq X$.

DEFINITION (CONFIGURATIONS). *Given an event structure* $I = (E, \#, \vdash)$ *a configuration of* I *is a subset* X *of* E *such that*

(i) X *is conflict-free:* $X \in Con_I$

(ii) *every event* $e \in X$ *has a proof in* X.

For this variant of Winskel's definition, one can still prove the representation theorem relating domains and stable event structures, the proof being essentially the same as in [20]. Then with each stable event structure one can associate an equivalent prime event structure. Conversely, given a flow event structure $S = (E, \#, \prec)$, we may define $\vdash_S$ as follows:

$$F \vdash_S e \Leftrightarrow_{\mathrm{def}} \begin{cases} F \cup \{e\} \in Con_S \quad \text{and} \\ e' \in F \Rightarrow e' \prec e \quad \text{and} \\ e' \prec e \Rightarrow \exists e'' \in F \; e' \, \natural \, e'' \end{cases}$$

It is trivial to check that $\mathcal{S}(S) = (E, \#, \vdash_S)$ is a stable event structure. Moreover it should be clear that S and $\mathcal{S}(S)$ are strongly equivalent since a sequence $e_1, \ldots, e_n$ is a proving sequence with respect to S if and only if it is a proving sequence with respect to $\mathcal{S}(S)$.

Given a stable event structure $I = (E, \#, \vdash)$ there does not necessarily exist a flow event structure S such that $S \equiv I$, that is $\mathcal{F}^\infty(S) = \mathcal{F}^\infty(I)$. Let us see this point in some details:

EXAMPLE 2 (FES $\subset$ SES). Let $\mathbf{S} = (\{a, b, c, d\}, \#, \vdash)$ be the stable event structure given by:

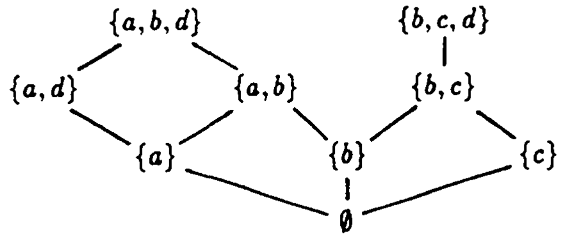
$a \# c$

$\emptyset \vdash a, b, c$

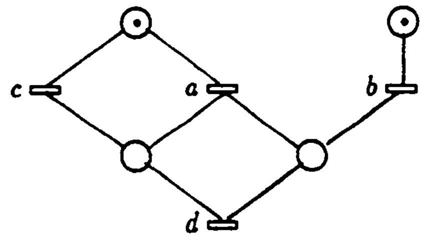
$\{a\} \vdash d$

$\{b, c\} \vdash d$

The domain of configurations of this structure is:



It is not hard to see, using the lemma 2.11, that this family of configurations cannot be that of a flow event structure, since we should have both $b \prec d$ and $b \not\prec d$. Note also that this structure may be represented by a net (which is not a flow net):



Appendix B: Bundle Event Structures and Safe Flow Nets.

In this section we show that safe flow nets concretely correspond to the *bundle event structures* introduced by Langerak [13]. Let us recall the definition:

DEFINITION (BUNDLE EVENT STRUCTURES). *A bundle event structure is a structure* $S = (E, \#, \mapsto)$ *where*

- E *is the denumerable set of events,*
- $\# \subseteq E \times E$ *is the symmetric conflict relation,*
- $\mapsto$ *is a relation from sets of events to events such that*

$$F \mapsto e \quad \& \quad e', e'' \in F \;\Rightarrow\; e' \# e''.$$

Intuitively, a "bundle" $F \mapsto e$ means that if e occurs, then at least one – and in fact exactly one – of the events of F must have occurred before. This is formalized in the definition of configurations. In [13] the notion of configuration of a bundle event structure is defined using an intermediary stable event structure. Here we introduce configurations by means of proving sequences:

DEFINITION (PROVING SEQUENCES). *Given* $S = (E, \#, \mapsto)$, a *proving sequence in* S *is a (finite or infinite) sequence* $e_1, \ldots, e_n, \ldots$ *of distinct non-conflicting events (i.e.* $i \neq j \Rightarrow e_i \neq e_j$ *and* $\neg(e_i \# e_j)$ *for all* i, j*) satisfying* $X \mapsto e_i \Rightarrow X \cap \{e_1, \ldots, e_{i-1}\} \neq \emptyset$.

A *configuration* of a bundle event structure S is a set X of events that can be enumerated as a proving sequence. As above we use the notations $\mathcal{F}(S)$ and $\mathcal{F}^\infty(S)$ respectively for the sets of finite configurations, and of all configurations of S, and we still use $S \equiv S'$ for $\mathcal{F}^\infty(S) = \mathcal{F}^\infty(S')$, where S and S' are event structures of any kind.

REMARK: unlike [13], we allow bundles of the form $\emptyset \mapsto e$. Clearly if this holds then the event e cannot occur in a configuration: in other words, e is semantically inconsistent, or self-conflicting.

Then we can easily show that any bundle event structure is strongly equivalent to another one without self-conflicts:

LEMMA. *For any bundle event structure* $S = (E, \#, \mapsto)$ *there exists* $S' = (E', \#', \mapsto')$ *such that* $S' \equiv S$ *and* $e \#' e' \Rightarrow e \neq e'$.

PROOF: we let $S' = (E, \#', \mapsto')$ where

$$e \#' e' \Leftrightarrow_{\text{def}} e \# e' \;\&\; e \neq e'$$

and

$$F \mapsto' e \Leftrightarrow_{\text{def}} F \mapsto e \text{ or } F = \emptyset \;\&\; e \# e$$

It is obvious that S and S' have the same proving sequences ❑

It is easy to see that any prime event structure can be turned into an equivalent bundle event structure:

LEMMA. *Let* $S = (E, \#, <)$ *be a prime event structure. Then the structure* $S' = (E, \#, \mapsto)$ *given by* $F \mapsto e \Leftrightarrow_{\text{def}} \exists e'.\, e' < e \;\&\; F = \{e'\}$ *is a bundle event structure such that* $S' \equiv S$.

Conversely, it is not the case that for any bundle event structure one can find a strongly equivalent prime event structure. The counter-example is more or less the same as the one showing that prime event structures are "less expressive" than flow event structures:

EXAMPLE 3 (PES $\subset$ BES). Let $\mathbf{B} = (\{e_0, e_1, e_2\}, \#, \mapsto)$ be the bundle event structure given by $e_0 \# e_1$ and $\{e_0, e_1\} \mapsto e_2$. Then there is no prime event structure S such that $S \equiv \mathbf{B}$.

Now we establish a correspondence from bundle event structures to Petri nets. This correspondence relates the configurations of the event structures with the firing sequences of the nets; therefore, due to the lemma above, we may deal with bundle event structures without self-conflicts. Given a bundle event structure $S = (E, \#, \mapsto)$, where the conflict relation is irreflexive, we define a net $\mathcal{N}(S)$ as follows: $\mathcal{N}(S) = (E, B_S, \Phi_S, \mu_S)$ where $B_S = B'_S \cup B''_S$ with(†):

$$B'_S =_{\text{def}} \bar{\Gamma}_S$$

and

$$B''_S =_{\text{def}} \{ (F, e) \mid F \mapsto e \}$$

The flow relation Φ_S is the least one such that

$$F \in B'_S \;\&\; e \in F \Rightarrow (F, e) \in \Phi_S$$

$$(G, e) \in B''_S \;\&\; e' \in G \Rightarrow (e', (G, e)) \in \Phi_S$$

$$(G, e) \in B''_S \Rightarrow ((G, e), e) \in \Phi_S$$

Finally the initial marking is given by:

$$\mu_S(b) =_{\text{def}} \begin{cases} 1 & \text{if } b \in B'_S \\ 0 & \text{otherwise} \end{cases}$$

(†) recall that $\bar{\Gamma}_S$ denotes the set of maximal (w.r.t. inclusion) pairwise conflicting sets of events. For any pairwise conlicting set of events F there exists $H \in \bar{\Gamma}_S$ such that $F \subseteq H$.

One can see that the net $\mathcal{N}(S)$ has a special shape. Let us call *s-regular flow net* any net $N = (E, B, \Phi, \mu_0)$ satisfying:

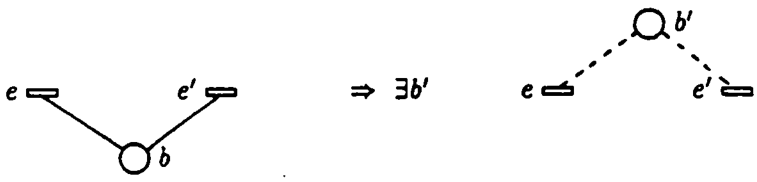
(i) a forward branched place is a choice place, that is:

$$\exists e.\ \phi(e,b) = 1 \ \Rightarrow\ (\phi(b,e') = 1 = \phi(b,e'') \ \Rightarrow\ e' = e'')$$

(ii) only choice places can be initially marked, with at most one token:

$$\mu_N(b) > 0 \ \Rightarrow\ \mu_N(b) = 1 \quad \& \quad \forall e.\ \phi(e,b) = 0$$

(iii) two (distinct) pre-events of a given place are in direct conflict:



that is, formally:

$$\forall e, e' \ \big(\exists b.\ \phi(e,b) = 1 = \phi(e',b)\big) \ \Rightarrow\ \big(\exists b'.\ \phi(b',e) = 1 = \phi(b',e')\big)$$

Then clearly for any bundle event structure S the net $\mathcal{N}(S)$ is an s-regular flow net. It should be obvious that an s-regular flow net is a regular flow net. Moreover such a net is safe, since any postcondition is a strong one.

PROPOSITION. *Let $S = (E, \#, \mapsto)$ be a bundle event structure, with $e \# e' \Rightarrow e \neq e'$. Then $\mathcal{N}(S)$ is a safe flow net such that $\mathcal{F}^\infty(\mathcal{N}(S)) = \mathcal{F}^\infty(S)$. More precisely a sequence $e_1, \ldots, e_n, \ldots$ of events is firable in $\mathcal{N}(S)$ if and only if it is a proving sequence in S.*

PROOF:

(i) let $e_1, \ldots, e_n, \ldots$ be a proving sequence of S. We show that the sequence $e_1, \ldots, e_n, \ldots$ may be fired in $\mathcal{N}(S)$, that is there exist markings $\mu_0 = \mu_S, \ldots, \mu_n, \ldots$ such that μ_{i-1} enables e_i for any i, and $\mu_{i-1} \xrightarrow{e_i} \mu_i$. Assume the contrary, and let k be the least index for which this is false, that is $\mu_0 = \mu_S \xrightarrow{e_1} \mu_1 \cdots \mu_{k-2} \xrightarrow{e_{k-1}} \mu_{k-1}$ is a firing sequence of $\mathcal{N}(S)$ and there is a precondition b of e_k such that $\mu_{k-1}(b) = 0$. There are two cases:

- if $b \in \bar{\Gamma}_S$ then $\mu_0(b) = 1$, and $e \in b \Rightarrow e \# e_k$. Since $e_1 \ldots e_n \ldots$ is made of distinct, non-conflicting events, we have $b \cap \{e_1, \ldots, e_{k-1}\} = \emptyset$, therefore $\mu_{k-1}(b) \geq \mu_0(b) > 0$ (in fact $\mu_n(b) = 1$ since for no e we have $(e,b) \in \Phi_S$), a contradiction.
- if $b = (F, e_k)$ with $F \mapsto e_k$ then $F \cap \{e_1, \ldots, e_{k-1}\} \neq \emptyset$. Therefore by definition of Φ_S there exists $i < k$ such that $\mu_{i+1}(b) > 0$. Since in this case $(b,e) \in \Phi_s \Rightarrow e = e_k$, and $e_k \notin \{e_{i+1}, \ldots, e_{k-1}\}$ we have $\mu_{k-1}(b) \geq \mu_{i+1}(b) > 0$, a contradiction.

(ii) now let $\mu_0 = \mu_S \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n \cdots$ be a firing sequence in $\mathcal{N}(S)$. We show that $e_1, \ldots, e_n, \ldots$ is a proving sequence of S:

- assume that there exist i and j such that $i < j$ and $e_i \# e_j$. Then there would exist $b \in \bar{\Gamma}_S$ such that $\{e_i, e_j\} \subseteq b$, hence $\phi_S(b, e_i) = 1 = \phi_S(b, e_j)$. But this is impossible since $\mathcal{N}(S)$ is a flow net, hence the precondition b cannot be used twice in a firing sequence. Therefore the sequence $e_1, \ldots, e_n, \ldots$ is made of distinct, non-conflicting events (there is no self-conflicting event).

• if $F \mapsto e_i$ then $b = (F, e_i)$ is a precondition of e_i, therefore $\mu_{i-1}(b) > 0$. It is not the case that $i - 1 = 0$. therefore there exists $j < i$ such that $(e_j, b) \in \Phi_S$, and this means $e_j \in F$, hence $F \cap \{e_1, \ldots, e_{i-1}\} \neq \emptyset$ $\square$

Conversely for any safe flow net one can build an equivalent bundle event structure. Let $N = (E, B, \Phi, \mu_N)$ be a safe flow net. We define a structure $\mathcal{B}(N)$ as follows: $\mathcal{B}(N) = (E, \#_N, \mapsto_N)$ where $\#_N$ is the semantical conflict of the net, that is

$$e \#_N e' \Leftrightarrow_{\text{def}} \forall X \in \mathcal{F}^{\infty}(N) \quad \{e, e'\} \not\subseteq X$$

and $F \mapsto_N e$ if and only if there exists $b \in B$ such that $\mu_N(b) = 0$ and $\phi(b, e) = 1$, and F is a maximal set of pairwise conflicting pre-events of b, that is F is maximal among the sets $G \subseteq E$ satisfying:

$$\begin{cases} e' \in G \Rightarrow \phi(e', b) = 1 \quad \text{and} \\ e', e'' \in G \Rightarrow e' \natural_N e'' \end{cases}$$

Clearly $\mathcal{B}(N)$ is a bundle event structure. Note that we have $\emptyset \mapsto_N e$ if there is a precondition b of e which is initially unmarked and which is not a postcondition of an event. In this case we also have $e \#_N e$.

PROPOSITION. *For any safe flow net N we have $\mathcal{F}^{\infty}(\mathcal{B}(N)) = \mathcal{F}^{\infty}(N)$. More precisely a sequence $e_1, \ldots, e_n, \ldots$ is firable in N if and only if it is a proving sequence in $\mathcal{B}(N)$.*

PROOF:

(i) we first show that if $\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n \cdots$ is a firing sequence of the net N then $e_1, \ldots, e_n, \ldots$ is a proving sequence of $\mathcal{B}(N)$:

• since N is a flow net we have $i \neq j \Rightarrow e_i \neq e_j$, and, by definition of the conflict relation, $\neg(e_i \#_N e_j)$.

• if $F \mapsto_N e_i$ then there exists $b \in B$ such that $\mu_0(b) = 0$ and $e \in F \Rightarrow \phi(e, b) = 1 = \phi(b, e_i)$. Since $\mu_{i-1}(b) > 0$ there exists $j < i$ such that $\phi(e_j, b) = 1$. We show that $e_j \in F$; assume the contrary: then there would exist $e \in F$ such that $\neg(e \#_N e_j)$. Therefore, by definition of the conflict relation, there would exist a firing sequence of N where e and e_j both occur. But this contradicts the lemma 3.2, since the flow net N is safe.

(ii) now let $e_1, \ldots, e_n, \ldots$ be a proving sequence of $\mathcal{B}(N)$. We show that there exist markings $\mu_0 = \mu_S, \ldots, \mu_n, \ldots$ such that μ_{i-1} enables e_i for any i, and $\mu_{i-1} \xrightarrow{e_i} \mu_i$, that is:

$$\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{n-1} \xrightarrow{e_n} \mu_n \cdots$$

is a firing sequence of N. Note that if such a sequence exists it is unique. Assume the contrary, and let k be the least index for which this is false, that is $\mu_0 = \mu_N \xrightarrow{e_1} \mu_1 \cdots \mu_{k-2} \xrightarrow{e_{k-1}} \mu_{k-1}$ is a firing sequence and there is a precondition b of e_k such that $\mu_{k-1}(b) = 0$. There are two cases:

• if $\mu_N(b) > 0$ then $i < k \Rightarrow \phi(b, e_i) = 0$ since otherwise $e_1, \ldots, e_n, \ldots$ would not be conflict-free, or would contain twice the same event (recall that $\phi(b, e') = 1 = \phi(b, e'') \Rightarrow e' \natural_N e''$, since N is a flow net). Therefore $\mu_{k-1}(b) \geq \mu_N(b) > 0$, a contradiction.

• otherwise, i.e. if $\mu_N(b) = 0$, by definition of $\mathcal{B}(N)$, there exists F such that $F \mapsto e_k$, therefore $F \cap \{e_1, \ldots, e_{k-1}\} \neq \emptyset$. Let $e_i \in F \cap \{e_1, \ldots, e_{k-1}\}$; then $\phi(e_i, b) = 1$, therefore $\mu_{i+1}(b) > 0$. The same argument as above shows that $i < j < k \Rightarrow \phi(b, e_j) = 0$, otherwise we would have $e_j \,\sharp_N\, e_k$, hence $\mu_{k-1}(b) \geq \mu_{i+1}(b) > 0$, a contradiction ◻

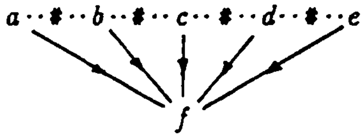
Two obvious consequences of the previous results are:

COROLLARY. *For any safe flow net N there exists an s-regular flow net N' such that $N \equiv N'$.*

COROLLARY (THIRD REPRESENTATION THEOREM). *A poset $(D, \leq)$ is a domain if and only if it is isomorphic to the family of configurations of some bundle event structure.*

To conclude this section we show that there exists a flow event structure, or equivalently a flow net, which not strongly equivalent to a bundle event structure, or equivalently a safe flow net. The example is provided by the net $\mathbf{W}$ of the section 3.1:

EXAMPLE 4 (BES ⊂ FES). Let $\mathbf{F} = (\{a, b, c, d, e, f\}, \#, \prec)$ be the flow event structure given by:



The family of configurations of $\mathbf{F}$ is the one of the net $\mathbf{W}$. Then it is not difficult to see that in a bundle event structure which would give this family of configurations the conflict relation should be $a \# b \# c \# d \# e$. But then the only possible bundles for f are among $\{a, b\} \mapsto f$, $\{b, c\} \mapsto f$, $\{c, d\} \mapsto f$ and $\{d, e\} \mapsto f$, and it is easy to see that any possible choice would miss some configuration of $\mathbf{F}$ (clearly adding inconsistent events would not help).

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Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique

ISSN 0249-6399